portance of their discovery of the notion of irrationality. They did not ignore it because it belonged to the thought-world rather than to the material world. The author stressed the absurdity of neglecting the study of pure, abstract principles, for we know not when they may lead to practical results. He cited many instances of practical truths being first made known by pure mathematics and later verified by experiment. In conclusion he expressed the hope that he had shown, in some measure, that pure mathematics is not simply, or mainly, a most useful tool for the astronomer, the engineer, the physicist, the mechanic, and others, but it has in itself a real *practical* side.

5. In his second paper Professor Veblen discussed a notation for odd and even permutations which is suitable for the theory of determinants and for various determinant-like expressions which occur in differential invariants. Notations for a quantity which is ± 1, according to the parity of a permutation, have been in use for a long time in books on algebra. An important advance was made by Ricci and Levi-Civita who, in their absolute differential calculus, introduced quantities $\epsilon_{ij\cdots k}$ which are 0 if there are repetitions in the subscripts and otherwise positive or negative according to parity of permutation. The importance of this was appreciated by certain writers on differential invariants and relativity as, for example, Lipka and Eddington. Another step was made simultaneously by Alexander and Murnaghan who found a set of quantities $\delta_{\alpha\beta\cdots\gamma}^{ij\cdots k}$ which generalize both the ϵ and the Kronecker δ and have decided advantages in flexibility over the previous notations. With the aid of these notations it is possible to make the proofs of theorems on determinants very short and luminous, and also to write a great many formulas explicitly in such a way as to make it easy to see their significance.

7. The customary method, according to Minkowski, of dealing with special relativity, involves the use of the rectangular hyperbola. A more natural mode of approach would be through that of the railroad time-table, and distance-time diagram. Using the result of the Michelson-Morley experiment to secure a new space unit and a new time unit for a set of new axes, no longer at right angles like the original axes, it is possible to set up the Lorentz transformation in an elegant manner, with simple geometrical construction. The method worked out in some detail by Schlesinger of Giessen, has not yet found its way into English and American texts.

W. W. RANKIN, Jr., *Secretary-Treasurer*.

THE SECOND ANNUAL MEETING OF LOUISIANA–MISSISSIPPI SECTION

The second annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Hotel Edwards, Jackson, Mississippi, March 20-21, 1925.

All sessions of the Association were well attended, particularly the Friday evening session at which about two hundred were present. Among those attending were the following members of the Association:

A. E. Babbitt, H. E. Buchanan, H. Fox, J. A. Hardin, G. L. Harrell, J. R. Hitt, Anna M. Howe, C. G. Latimer, A. C. Maddox, W. M. Maiden, B. E. Mitchell, I. C. Nichols, R. L. O'Quinn, S. T. Sanders, J. M. Sharp, Nathaniel Smylie, Mary C. Spencer, W. B. Stokes, Robert Torrey, Norma Touchstone, B. M. Walker, S. P. Walker.

Both the Friday evening and the Saturday morning meetings were presided over by the regular chairman of the Section, Dr. B. M. Walker, then professor of mathematics at Mississippi A. &. M College, now president of the institution. Dinner was enjoyed together Friday evening. At the business session Saturday morning, the following officers were elected for 1925–1926:

Chairman, H. E. BUCHANAN, Tulane University,

Vice-Chairman, H. FOX, A. & M. College of Mississippi,

Secretary-Treasurer, I. C. NICHOLS, Louisiana State University. The next meeting was set for March, 1926, in New Orleans with Tulane University and Newcomb College.

Before taking up the regular program, Dr. Walker gave a brief discussion of a certain triple infinity of points. The following papers were then presented:

1. "Other worlds than ours," Professor F. R. MOULTON of the University of Chicago. An illustrated lecture.

2. "A comparative study of Euclidean and non-Euclidean hypotheses with emphasis on the logical aspects," Professor S. T. SANDERS, L. S. U.

3. "Some problems in projective geometry," Dr. ANNA M. HOWE, Newcomb College of Tulane University.

4. "On a certain binary quadratic congruence," Dr. C. G. LATIMER, Tulane University.

5. "Ballistics," Professor MOULTON, University of Chicago.

6. "The orthocentric quadrilateral," Professor R. L. O'QUINN, L. S. U.

7. "The mathematical status of the present day freshman," a discussion led by Professor ROBERT TORREY, University of Mississippi.

8. "Popular fallacies in actuarial science," Mr. A. E. BABBITT, Actuary, Lamar Life Insurance Company.

All papers were well presented and closely followed by those present. Special mention should be made of Dr. Moulton's two lectures, both of which were particularly attractive and profitable.

I. C. NICHOLS, *Secretary-Treasurer.*

EUCLIDEAN INVARIANTS OF SECOND DEGREE CURVES[1]

By C. C. MACDUFFEE, Ohio State University

1. Introduction. Although most works on coördinate geometry attempt to classify second degree curves into types by means of their invariants, the writer knows of no classification which is complete. A complete classification must not only separate the curves into ellipses, parabolas, hyperbolas, parallel lines and intersecting lines, but it must also pick out the imaginary loci from the real, and it must characterize by invariants the parameters in the canonical forms. The semi-axes of the ellipse and hyperbola, the latus rectum of the parabola, the angle between the intersecting lines and the distance between the parallel lines are invariants, and a complete classification of second degree curves must express these invariants in terms of a fundamental system.

The text by Bôcher[2] gives a treatment which is as good as any which the writer has found. Bôcher uses the three well-known invariants and by means of them classifies the second degree curves into types and separates the real curves from the imaginary in every case but one, namely the parallel line case. He also characterizes some of the parameters. He recognizes the fact that the three invariants at hand are insufficient to separate the case of real parallel lines from the case of imaginary parallel lines, and makes no further attempt to effect this separation on the basis of invariants.

Since the three invariants ordinarily used are insufficient completely to classify second degree curves, the idea suggests itself that perhaps there are other invariants or covariants of the second degree polynomial under euclidean transformations than those heretofore recognized. In the present article the Lie theory of continuous groups is brought into play to determine the exact number of functionally independent invariants and covariants. It is found that there are three invariants and two covariants, one other in addition to the given polynomial—a fact which seems to have escaped attention. A certain second degree polynomial Θ is found to serve as the missing covariant, and a complete system of invariants and covariants is exhibited.

This new covariant Θ is found to fill in an interesting manner the gap noticed by Bôcher. Its degree is invariant, and $\Theta = constant$ is a necessary and sufficient

[1] Read before the Ohio Section of the Association April 2, 1926.

[2] M. Bôcher, *Plane Analytic Geometry*, Henry Holt and Co., 1915, pp. 176–188.

condition that we have the parallel line case. This constant is of course an invariant for this case. The lines are real and distinct, real and coincident, or imaginary according as this constant value of Θ is negative, zero, or positive, and the distance between the lines is determined by Θ together with another invariant.

In §7 the general second degree equation is reduced to canonical forms. By using the methods of invariant theory, this reduction is made with a minimum of calculation. The parameters are characterized in terms of invariants.

Since our three invariants and two covariants form a Lie complete system, every invariant magnitude connected with a conic is a function of them, and so is the equation of every covariant curve. In particular, the equation of the director circle is $\Theta=0$. The eccentricity is an irrational invariant, a root of a fourth degree equation whose coefficients are invariants. Simple rational covariants of the hyperbola give the asymptotes and the conjugate hyperbola, while the axes of a central conic are given by a more complicated covariant. These examples serve to show the significance of a complete system of covariants.

The application of these methods to plane cubics and systems of conics, as well as to surfaces in space, should prove interesting and not too difficult.

2. The Lie theory of euclidean invariants.[1] We shall take as the group of euclidean transformations

$$x=x'\cos\omega-y'\sin\omega-\alpha\ ,$$
$$y=x'\sin\omega+y'\cos\omega-\beta\ ,$$

where ω, α, β are real parameters. From these we have

$$\begin{aligned} x'&=x\cos\omega+y\sin\omega+(\alpha\cos\omega+\beta\sin\omega)\ ,\\ y'&=-x\sin\omega+y\cos\omega-(\alpha\sin\omega-\beta\cos\omega)\ . \end{aligned} \tag{1}$$

The values $\omega_0=\alpha_0=\beta_0=0$ determine the identical transformation.

Let $f(x, y)$ be a real analytic function. We have $f(x', y')=F(\omega, \alpha, \beta), f(x, y)=F(\omega_0, \alpha_0, \beta_0)$. For sufficiently small values of ω, α, β we have by Taylor's formula

$$F(\omega,\alpha,\beta)=F(\omega_0,\alpha_0,\beta_0)+\sum_{i=1}^{\infty}\frac{1}{i!}\left(\omega\frac{\partial}{\partial\omega_0}+\alpha\frac{\partial}{\partial\alpha_0}+\beta\frac{\partial}{\partial\beta_0}\right)^i F(\omega_0,\alpha_0,\beta_0) \tag{2}$$

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[1] Lie-Scheffers, *Vorlesungen über kontinuierliche Gruppen*, Teubner, 1893, p. 100 and p. 716.
A. Cohen, *An Introduction to the Lie Theory of One-parameter Groups*, Heath, 1911, p. 211 *et seq.*
L. E. Dickson, "Differential equations from the group standpoint," *Annals of Mathematics*, vol. 25, no. 4, June, 1924.

where, as usual, we understand that

$$\left(\omega\frac{\partial}{\partial\omega_0}+\alpha\frac{\partial}{\partial\alpha_0}+\beta\frac{\partial}{\partial\beta_0}\right)F(\omega_0,\alpha_0,\beta_0)=\frac{\partial F(\omega,\alpha,\beta)}{\partial\omega}\Big]_{\omega_0,\alpha_0,\beta_0}\omega$$
$$+\frac{\partial F(\omega,\alpha,\beta)}{\partial\alpha}\Big]_{\omega_0,\alpha_0,\beta_0}\alpha+\frac{\partial F(\omega,\alpha,\beta)}{\partial\beta}\Big]_{\omega_0,\alpha_0,\beta_0}\beta\,.$$

Now

$$\frac{\partial F(\omega,\alpha,\beta)}{\partial\omega}\Big]_{\omega_0,\alpha_0,\beta_0}=\frac{\partial f(x',y')}{\partial x'}\frac{\partial x'}{\partial\omega}\Big]_{\omega_0,\alpha_0,\beta_0}+\frac{\partial f(x',y')}{\partial y'}\frac{\partial y'}{\partial\omega}\Big]_{\omega_0,\alpha_0,\beta_0}$$
$$=\frac{\partial x'}{\partial\omega}\Big]_{\omega_0,\alpha_0,\beta_0}\frac{\partial f(x,y)}{\partial x}+\frac{\partial y'}{\partial\omega}\Big]_{\omega_0,\alpha_0,\beta_0}\frac{\partial f(x,y)}{\partial y}\,.$$

Following Lie, we define the operator

$$U_\omega\equiv\frac{\partial x'}{\partial\omega}\Big]_{\omega_0,\alpha_0,\beta_0}\frac{\partial}{\partial x}+\frac{\partial y'}{\partial\omega}\Big]_{\omega_0,\alpha_0,\beta_0}\frac{\partial}{\partial y}=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\,.$$

Similarly we have

$$U_\alpha\equiv\frac{\partial x'}{\partial\alpha}\Big]_{\omega_0,\alpha_0,\beta_0}\frac{\partial}{\partial x}+\frac{\partial y'}{\partial\alpha}\Big]_{\omega_0,\alpha_0,\beta_0}\frac{\partial}{\partial y}=\frac{\partial}{\partial x}\,,$$

$$U_\beta\equiv\frac{\partial x'}{\partial\beta}\Big]_{\omega_0,\alpha_0,\beta_0}\frac{\partial}{\partial x}+\frac{\partial y'}{\partial\beta}\Big]_{\omega_0,\alpha_0,\beta_0}\frac{\partial}{\partial y}=\frac{\partial}{\partial y}\,.$$

We define the operator

$$U\equiv\omega U_\omega+\alpha U_\alpha+\beta U_\beta,$$

which is completely defined by transformation (1). Evidently we have

$$\left(\omega\frac{\partial}{\partial\omega_0}+\alpha\frac{\partial}{\partial\alpha_0}+\beta\frac{\partial}{\partial\beta_0}\right)F(\omega_0,\alpha_0,\beta_0)=Uf(x,y)\,.$$

In a similar manner we can demonstrate that

$$\left(\omega\frac{\partial}{\partial\omega_0}+\alpha\frac{\partial}{\partial\alpha_0}+\beta\frac{\partial}{\partial\beta_0}\right)^i F(\omega_0,\alpha_0,\beta_0)=U^if(x,y)$$

so that (2) can be written

$$f(x',y')=f(x,y)+Uf(x,y)+\frac{1}{2!}U^2f(x,y)+\frac{1}{3!}U^3f(x,y)+\cdots \qquad (3)$$

In particular when we take x' and y' successively for $f(x', y')$ we have group (1) in the form

$$x'=x+Ux+\frac{1}{2!}U^2x+\frac{1}{3!}U^3x+\cdots,$$

$$y'=y+Uy+\frac{1}{2!}U^2y+\frac{1}{3!}U^3y+\cdots.$$

Thus the operator U completely determines the group (1) for sufficiently small values of ω, α, β. Lie calls U the *symbol* of transformation (1), and U_ω, U_α, U_β the *generators* of (1).

Let $f(x, y)$ be an invariant under transformations (1), *i. e.*, a function such that $f(x', y') \equiv f(x, y)$ identically in ω, α, β. From (3) it is evident that we must have

$$Uf = \omega U_\omega f + \alpha U_\alpha f + \beta U_\beta f \equiv 0 ,$$

that is,

$$U_\omega f = 0 , \qquad U_\alpha f = 0 , \qquad U_\beta f = 0 .$$

Since $Uf \equiv 0$ implies $U^i f \equiv 0$, this condition is also sufficient for small values of the parameters.

3. The Lie group for a second degree polynomial. The transformation (1) induces upon the coefficients of the real polynomial

$$\Phi \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c \tag{4}$$

the transformation

$$\begin{aligned}
a' &= c \cos^2\omega + 2h \sin\omega \cos\omega + b \sin^2\omega , \\
h' &= -a \sin\omega \cos\omega + h(\cos^2\omega - \sin^2\omega) + b \sin\omega \cos\omega , \\
b' &= a \sin^2\omega - 2h \sin\omega \cos\omega + b \cos^2\omega , \\
g' &= -a\alpha \cos\omega - h\beta \cos\omega - h\alpha \sin\omega - b\beta \sin\omega + g \cos\omega + f \sin\omega , \\
f' &= a\alpha \sin\omega + h\beta \sin\omega - h\alpha \cos\omega - b\beta \cos\omega - g \sin\omega + f \cos\omega , \\
c' &= a\alpha^2 + 2h\alpha\beta + b\beta^2 - 2g\alpha - 2f\beta + c .
\end{aligned} \tag{5}$$

These transformations together with (1) form a group whose generators are found to be

$$\begin{aligned}
U_\omega &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + 2h\frac{\partial}{\partial a} + (b-a)\frac{\partial}{\partial h} - 2h\frac{\partial}{\partial b} + f\frac{\partial}{\partial g} - g\frac{\partial}{\partial f} , \\
U_\alpha &= \frac{\partial}{\partial x} - a\frac{\partial}{\partial g} - h\frac{\partial}{\partial f} - 2g\frac{\partial}{\partial c} , \\
U_\beta &= \frac{\partial}{\partial y} - h\frac{\partial}{\partial g} - b\frac{\partial}{\partial f} - 2f\frac{\partial}{\partial c} .
\end{aligned} \tag{6}$$

From the Lie theory we know that the differential equations $U_\omega = 0$, $U_\alpha = 0$, $U_\beta = 0$ form a complete system,[1] since they are the generators of a group. The three equations in eight variables have $8 - 3 = 5$ functionally independent solutions, which are the 5 invariants and covariants of Φ, and we know that all other invariants and covariants are functions of these.

[1] Goursat-Hedrick, *Mathematical Analysis*, vol. 2, part 2, Ginn, 1917, p. 267.

If in (6) we delete the terms in x and y, we obtain the generators of the group (5) alone. The corresponding differential equations form a complete system with $6-3=3$ functionally independent solutions, namely the 3 invariants of Φ. Thus a complete system of invariants and covariants of Φ consists of 3 invariants and 2 covariants.

4. A complete system of invariants and covariants. While it is possible to solve the differential equations: $U_\omega=0$, $U_\alpha=0$, $U_\beta=0$ and thus obtain all our invariants and covariants, it is not necessary in this case to do so. The following functions are annihilated by each of the operators U_ω, U_α, U_β and hence are invariants and covariants:

Φ, $\Lambda=a+b$, $\Sigma=ab-h^2$, $\Delta=abc-af^2-bg^2-ch^2+2fgh$, $\Theta=(ab-h^2)(x^2+y^2)+2(bg-hf)x+2(af-hg)y+ac+bc-g^2-f^2$.

If we can now show that these 5 functions are functionally independent, we shall have shown that they form a Lie complete system of invariants and covariants. In particular they are independent when $b=g=c=0$, for in this case the Jacobian

$$\frac{\partial(\Lambda,\Sigma,\Delta,\Phi,\Theta)}{\partial(x,y,a,f,h)}$$

does not vanish identically. We will spare the reader the calculation.

5. Invariance of the degree of a covariant. The covariants Φ and Θ may be of degree 2, degree 1, degree 0, or vanish identically. The degree of each covariant is an arithmetic invariant. It is obvious, since transformations(1) are linear, that the degree of a covariant cannot be raised. It follows then that it cannot be lowered, for this would imply that it could be raised by the inverse transformation which again is linear.

6. Geometric invariants. If we multiply (4) through by a real number $k\neq 0$, we do not alter the locus of $\Phi=0$. The invariants and covariants are changed, however, becoming

$$k\Lambda,\ k^2\Sigma,\ k^3\Delta,\ k\Phi,\ k^2\Theta\ . \tag{7}$$

The invariants cannot, then, measure geometric magnitudes, for they do not always have the same values for congruent curves. We shall call them *algebraic* invariants and covariants of *weights* 1, 2, 3, 1, 2, respectively.

It is easy to form invariants, *e. g.* $\frac{\Sigma}{\Lambda^2}$, $\frac{\Delta}{\Lambda^3}$, $\frac{\Delta^2}{\Sigma^3}$, which actually maintain a fixed value for all congruent curves. These we shall call *geometric* invariants. We see that there are just two independent geometric invariants.

The invariants (7) may be used in classifying curves, however, for since $k\neq 0$ their vanishing or non-vanishing is an invariant property of the curve. Moreover, the signs of invariants of even weight, as Σ and $\Lambda\Delta$, are unalterable and

may be used in separating types of curves. It is impossible to classify curves according to the sign of Δ.[1]

7. Reduction of the second degree equation. We have seen that the degree of Φ is invariant. If Φ is of degree 1, the locus of $\Phi=0$ is a straight line. If Φ is of degree 0, there is no locus. If Φ has no degree (vanishes identically), the locus is the entire plane. We shall henceforth assume that Φ is of degree 2, *i. e.*, that $(a, h, b)\neq(0, 0, 0)$. This is true when and only when the invariant $\Lambda^2-2\Sigma=a^2+b^2+2h^2$ is positive.

From (5) we see that

$$h'=2h\cos 2\omega-(a-b)\sin 2\omega\ .$$

It is evidently possible under all circumstances to choose ω so that $h'=0$. If $a-b\neq 0$, ω is any value satisfying

$$\tan 2\omega=\frac{2h}{a-b}\ .$$

If $a-b=0$, $\omega=\pi/4$ will make $h'=0$ irrespective of whether also $h=0$ or not. Thus every equation $\Phi=0$ can be reduced (dropping primes) to the form

$$ax^2+by^2+2gx+2fy+c=0\ , \qquad (a,b)\neq(0,0)\ . \tag{8}$$

We now divide our analysis into cases.

Case I. $\Sigma\neq 0$. Since $h=0$, we now have $\Sigma=ab\neq 0$. If we apply to (8) the transformation

$$x=x'-\frac{g}{a}\ , \qquad y=y'-\frac{f}{b}\ ,$$

we obtain

$$ax^2+by^2+c=0\ , \qquad ab\neq 0\ . \tag{9}$$

If we set $h=g=f=0$ in Δ, we find that $\Delta=abc$, so that $\Delta=0$ when and only when $c=0$.

Case I A. $\Sigma\neq 0$, $\Delta\neq 0$. We have

$$ax^2+by^2+c=0\ , \qquad abc\neq 0\ ,$$

which is readily reducible to the generic canonical form

$$\frac{x^2}{\xi}+\frac{y^2}{\eta}=1\ , \qquad \xi\eta\neq 0\ . \tag{10}$$

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[1] A certain well-known American text proposes to do exactly this.

The parameters ξ and η are irrational geometric invariants. We have $a=\eta$, $b=\xi$, $c=-\xi\eta$, $h=f=g=0$, and hence

$$\Lambda=\xi+\eta\,, \qquad \Sigma=\xi\eta\,, \qquad \Delta=-\xi^2\eta^2\,. \tag{11}$$

The quotient $-\Sigma^2/\Delta=1$ is an algebraic invariant of weight 1, so we may characterize ξ and η by geometric invariants as follows:

$$\xi+\eta=\frac{-\Lambda\Delta}{\Sigma^2}\,, \qquad \xi\eta=\frac{\Delta^2}{\Sigma^3}\,.$$

Hence ξ and η are the roots of

$$\Sigma^3u^2+\Lambda\Sigma\Delta u+\Delta^2=0 \tag{12}$$

Each coefficient is of weight 6, so ξ and η are geometric invariants. The values are both real, for the discriminant

$$\Delta^2(\Lambda^2-4\Sigma)=\Delta^2[(a-b)^2+4h^2]$$

can never be negative.

This is as far as we need to go, but it is customary to subdivide (10) into three cases according as ξ and η are both positive, both negative, or opposite in sign. Since $\xi\eta$ has the sign of Σ, they are opposite in sign when and only when $\Sigma<0$, and we have the hyperbola. When $\Sigma>0$ and $\Lambda\Delta<0$, both roots of (12) are positive and we have the ellipse. When $\Sigma>0$ and $\Lambda\Delta>0$, there is no locus. We cannot have $\Lambda\Delta=0$ when $\Sigma>0$.

Case I B. $\Sigma\neq0$, $\Delta=0$. Now (9) assumes the form

$$ax^2+by^2=0\,, \qquad ab\neq0\,. \tag{13}$$

The coefficients a and b are algebraic invariants of weight 1, the roots of the quadratic equation

$$u^2-\Lambda u+\Sigma=0 \tag{14}$$

Since only the ratio $a:b$ is of geometrical significance, we shall consider the symmetrical equation (13) as our canonical form rather than divide through by a or b.

The discriminant $\Lambda^2-4\Sigma=(a-b)^2$ of (14) is never negative, so the roots are real. When $\Sigma<0$ the roots are opposite in sign and (13) consists of two intersecting lines. When $\Sigma>0$ the roots are like in sign and the graph consists of a single point.

If we denote by θ one of the angles between the lines (13) when $\Sigma<0$, we find that $\tan\theta$ is a geometric invariant defined by the equation

$$\Lambda^2\tan^2\theta+4\Sigma=0\,.$$

Case II. $\Sigma=0$. Since $h=0$, we now have $\Sigma=ab=0$ and either $a\neq 0$, $b=0$ or $a=0$, $b\neq 0$. The second alternative can be reduced to the first by the transformation $x=y'$, $y=-x'$. Hence (8) can be written

$$x^2+2gx+2fy+c=0\ .$$

The transformation $x=x'-g$, $y=y'$ gives

$$x^2+2fy+c=0\ . \tag{15}$$

Let us set $b=h=g=0$, $a=1$ in Δ. We have $\Delta=-f^2$, so that $\Delta=0$ when and only when $f=0$.

Case II A. $\Sigma=0$, $\Delta\neq 0$. We now have $f\neq 0$ so that the transformation

$$x=x'\ , \qquad y=y'-\frac{c}{2f}\ ,$$

is valid, reducing (15) to the canonical form

$$x^2=4py\ , \qquad p\neq 0\ . \tag{16}$$

The parameter p in (16) is an irrational invariant. We have for (16) $a=1$, $f=-2p$ and all other coefficients 0, so that $\Lambda=1$, $\Delta=-4p^2$. Since $\Lambda\neq 0$ for the canonical form, we know that it is different from zero for the entire case. Similarly we know that $\Delta/\Lambda^3<0$ for the entire case. Thus we may take as p either one of the real non-zero geometric invariants defined by the equation

$$4\Lambda^3p^2+\Delta=0\ .$$

Case II B. $\Sigma=\Delta=0$. We have $f=0$ so that (15) assumes the canonical form

$$x^2+c=0\ . \tag{17}$$

For (17) we have $\Lambda=1$, $\Sigma=\Delta=0$. The invariants are therefore insufficient to characterize this case. But the covariant Θ meets our needs in a peculiar and interesting manner. We have for (17) $\Theta=c$. Since the degree of Θ is invariant we see that Θ reduces to a constant whenever $\Sigma=\Delta=0$. Conversely, since

$$\Delta=c(ab-h^2)-f(af-hg)-g(bg-hf)\ ,$$

it follows that when Θ reduces to a constant, $\Sigma=\Delta=0$. Thus the condition that Θ reduce to a constant characterizes this case.[1]

Evidently (17) represents coincident lines when and only when $\Theta\equiv 0$, parallel lines when and only when $\Theta=constant<0$, and no locus when and only when $\Theta=constant>0$.

The distance d between the lines (17) is given by $d^2=-4\Theta/\Lambda^2$.

[1] *Cf.* B. Niewenglowski, *Cours de géométrie analytique*, Gauthier-Villars, 1894, vol. 1, p. 282. Niewenglowski obtains the constant term of Θ and recognizes that it is invariant when $\Sigma=\Delta=0$, and uses it to obtain the distance between the parallel lines.

8. Further invariants. Every geometric magnitude which is intrinsic to a curve—*i.e.*, is not dependent upon the particular coördinates used—is obviously an invariant under transformation of coördinates. A magnitude which is defined for all curves of a canonical set of the family $\Phi=0$ is given by a formula which is an invariant function of the coefficients of Φ.

We shall give two illustrations.

The area of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is πab. We have for this canonical form $\Sigma=a^2b^2$, $\Delta=-a^4b^4$. Hence $\Delta^2/\Sigma^3=a^2b^2$ is a geometric invariant, and so for every ellipse the area is $\pi\Delta\Sigma^{-3/2}$. The formula is based upon the above canonical form and has no interpretation for curves not reducible to this form.

The eccentricity e of the ellipses and hyperbolas (10) as customarily defined is given by

$$e^2=\frac{\xi-\eta}{\xi}, \qquad \xi>0, \qquad \xi\geqq\eta.$$

Hence $e^2-2=-(\xi+\eta)/\xi$, so that

$$\frac{e^4}{(e^2-2)^2}=\frac{(\eta-\xi)^2}{(\eta+\xi)^2}.$$

But from (11),

$$(\eta-\xi)^2=(\eta+\xi)^2-4\xi\eta=\frac{\Lambda^2\Delta^2}{\Sigma^4}-\frac{4\Sigma\Delta^2}{\Sigma^4},$$

$$\frac{e^4}{(e^2-2)^2}=\frac{\Lambda^2-4\Sigma}{\Lambda^2}.$$

Therefore e is a root of the equation

$$\Lambda^2e^4=(\Lambda^2-4\Sigma)(e^2-2)^2,$$

which is equivalent to[1]

$$(e^2-1)^2\Sigma+(e^2-1)(\Lambda^2-2\Sigma)+\Sigma=0.$$

9. Covariant curves. The equation of every covariant curve can be written in terms of the five fundamental covariants. Our simplest example is the locus of all points from which orthogonal tangents can be drawn to the conic. The equation of this locus is $\Theta=0$. For central conics (10) it is the director circle,[2] for the parabola it is the directrix.

A covariant curve of the hyperbola

$$b^2x^2-a^2y^2-a^2b^2=0$$

is evidently the conjugate hyperbola

$$b^2x^2-a^2y^2+a^2b^2=0.$$

For this case

$$\Phi=b^2x^2-a^2y^2-a^2b^2, \qquad \Sigma=-a^2b^2, \qquad \Delta=a^4b^4.$$

Hence $\Delta/\Sigma=-a^2b^2$ is of the same weight as Φ, and

$$\Phi-\frac{2\Delta}{\Sigma}=0$$

is the equation of the conjugate hyperbola. Since these expressions are covariants, we know that the hyperbola conjugate to an arbitrary hyperbola $\Phi=0$ has the equation[3]

$$\Sigma\Phi-2\Delta=0.$$

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[1] See S. L. Loney, *The Elements of Coördinate Geometry*, Macmillan, 1897, p. 344. Loney obtains many interesting results without calling attention to their invariant nature.

[2] Loney, p. 365.

[3] Loney, p. 329.

The asymptotes of a hyperbola evidently constitute a covariant curve. By an argument similar to the above, we conclude that their equation is given by

$$\Sigma\Phi-\Delta=0.$$

To find the equations of the axes of a central conic in covariant form is a problem of somewhat more complexity. For the canonical form (10) we have

$$\Phi=\eta x^2+\xi y^2-\xi\eta\ , \qquad \Theta=\xi\eta(x^2+y^2)-\xi\eta^2-\xi^2\eta\ ,$$
$$\Lambda=\xi+\eta\ , \qquad \Sigma=\xi\eta\ , \qquad \Delta=-\xi^2\eta^2\ .$$

Hence

$$\eta x^2+\xi y^2=\Phi-\Delta/\Sigma\equiv G$$

is a covariant of weight 1, and

$$\eta^2x^2+\xi^2y^2=\Lambda\Phi-\Theta\equiv H$$

is of weight 2. We find from these equations that

$$x^2y^2=\frac{\xi\eta G^2-\eta GH-\xi GH+H^2}{-\xi\eta(\xi-\eta)^2}=\frac{\Sigma G^2-\Lambda GH+H^2}{-\Sigma[\Lambda^2-4\Sigma]}\equiv M\ .$$

For the canonical form (10) the axes are the lines $x=0$ and $y=0$. For the canonical form the covariant M is a perfect square and its square root factors into linear factors. If we transform back from (10) to (4), the lines $x=0$ and $y=0$ are transformed into the axes and the properties of M of factoring are not destroyed. If therefore we form M from the original coefficients and variables it will be a perfect square, and this square root will factor into linear factors which, equated to zero, will give the axes of the conic.

For central conics the factor Σ in the denominator is not zero. The factor $\Lambda^2-4\Sigma$ tells us that no determinate axes will be obtained when $\Lambda^2-4\Sigma=0$. This was to be expected, since the invariant $\Lambda^2-4\Sigma$ vanishes when and only when $\Phi=0$ is a circle.

Let it be required to find the equations of the axes of the hyperbola $2xy-1=0$. We have

$$\Phi=2xy-1\ , \qquad \Theta=-x^2-y^2\ , \qquad \Lambda=0\ , \qquad \Sigma=-1\ , \qquad \Delta=1\ ,$$
$$G=2xy\ , \qquad H=x^2+y^2\ , \qquad M=\tfrac{1}{4}(x^2-y^2)^2\ .$$

The axes are therefore $x+y=0$ and $x-y=0$.

THE ELEMENTARY CHARACTER OF CERTAIN MULTIPLE INTEGRALS CONNECTED WITH FIGURES BOUNDED BY PLANES AND SPHERES[1]

By PHILIP FRANKLIN, Massachusetts Institute of Technology

1. Introduction. In the classical problem of calculating the area, or volume, cut out of a sphere by a square prism with one diameter of the sphere as an axis, one encounters certain integrals, which, although reducible to elementary integrals by appropriate substitutions, at first sight appear to be elliptic integrals of a non-elementary character. This leads to the question as to how far one can go in raising problems about figures formed of spheres and planes without meeting other than elementary integrals. In this paper we shall investigate this question, and, by fairly simple methods shall justify the following conclusions. *For any surface or volume obtained from a figure formed entirely of spheres and planes, the content, coördinates of the center of gravity, and moment of*

[1] Presented to the American Mathematical Society, October 31, 1925.

inertia about any axis are elementary functions of the parameters determining the figure. The parameters referred to here are either the coefficients of the equations of the planes and spheres, or the coördinates of a set of points sufficient to determine the figure, or a group of lengths and angles of this character. The statement just made does not necessarily imply the elementary character of the integrals used to obtain the contents, moments, and moments of inertia, since these integrals are definite. Our methods of proof, however, show that there is *at least one method of so setting up these integrals that all the intermediate indefinite integrals will be elementary.* One other property of our solution may be noticed. No exponential or logarithmic terms appear, as all the quantities referred to in our theorem involve only algebraic, trigonometric, or inverse trigonometric functions and their combinations.

2. The dissection of surfaces. We shall reduce the treatment of our problem in the general case to that for a small number of simpler figures, into which every figure of the type we are considering may be decomposed. We begin with the case of surfaces, for which we have:

THEOREM I. *Any surface obtained from a figure composed entirely of spheres and planes may be decomposed into a number of plane right triangles, spherical right triangles, circular sectors on a plane and circular sectors on a sphere.*

For, the plane parts of such a figure are curvilinear polygons, whose sides are straight lines or arcs of circles. To break up such a curvilinear polygon, we select any point O in its plane. We join this with each vertex of the polygon by a straight line, and also draw all the tangents to the circular sides of our polygon which pass through O. We mark in the intersections of all the lines just drawn with the polygon, and regard them as vertices in our later steps. Since we are only considering figures made up of a finite number of spheres and planes, the number of vertices is finite. Taking them in order on the curvilinear polygon, we denote them by $A_1, A_2, \cdots A_n$. Our original figure may now be looked upon as the sum of the figures $O\,A_i\,A_{i+1}$, taken with proper signs.[1] Each such figure is either a triangle, which is reduced to two right triangles by drawing in an altitude, or it is bounded by the circular arc A_iA_{i+1} and the straight lines OA_i, OA_{i+1}. In the latter case, we join B_i, the center of the circle containing arc A_iA_{i+1}, with A_i and A_{i+1}, and also draw in $\overline{A_iA}_{i+1}$, the chord of arc $\overset{\frown}{A_iA}_{i+1}$. We then have:

$$O\,A_iA_{i+1} = O\,\overline{A_iA}_{i+1} \pm (B_i\,A_iA_{i+1} - B_i\,\overline{A_iA}_{i+1}).$$

The triangles OA_iA_{i+1} and $B_iA_iA_{i+1}$ are divided by altitudes, and $B_i\overset{\frown}{A_iA}_{i+1}$ is a sector. Thus we have shown how to break up all the plane parts of our figure into right triangles and sectors.

[1] By considering our areas as positive or negative in accordance with the sense of rotation along the boundary, we could eliminate the indeterminacy of sign here, as later. We have used positive areas, with indeterminate signs instead of this convention in the interest of simplicity.

For the spherical parts of our figure, we may use a similar method of reduction, with the straight lines and circles replaced by great circles and small circles respectively. We thus reduce these parts, or curvilinear polygons on a sphere, $A_1, A_2, \cdots A_n$ to a set of sectors on a sphere, $B_i\overset{\frown}{A_iA_{i+1}}$, and a number of right spherical triangles.

3. The dissection of volumes. For the reduction of volumes, we have an analogous theorem.

THEOREM II. *Any volume obtained from a figure composed entirely of spheres and planes may be decomposed into a number of pyramids, with right triangles as bases, cones with circular sectors as bases, spherical pyramids with right spherical triangles as bases and spherical cones with circular sectors on the sphere as bases.*

To prove this, we select any point in space O. We join this by planes and conical surfaces with vertex at O to all the straight line segments or circular arcs in which two of the plane or spherical surfaces intersect. We also draw tangent cones from O to the spheres of the figure. These surfaces enable us to regard any volume obtained from our figure as the sum, with suitable signs, of a number of conical figures, each of which has a base either in a single plane, or on a single sphere.

For any such cone in which the base is plane, we apply the reduction of section 2 to the base, reducing it to right triangles and circular sectors. On joining the boundaries of these figures to the vertex, we reduce this cone to a number of pyramids with right triangles as bases, and a number of cones with circular sectors as bases.

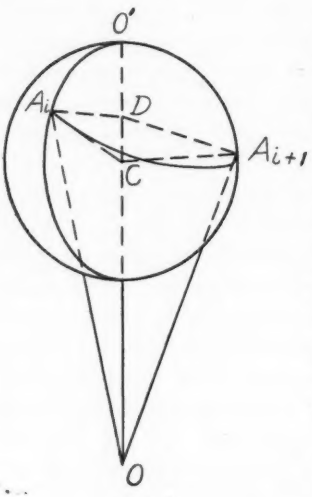

FIG. 1.

For a conical figure in which the base is spherical, we proceed as follows. The base is a spherical curvilinear polygon, on a single sphere, say with center at C. Join OC (Fig. 1) and prolong it to intersect the sphere at O'. Either intersection may be used. Now join O' with each of the vertices of the curvilinear polygon by great circles on the sphere. The tangent great circles to the sides which pass through O' are also drawn. We regard all the points of intersection of these great circles with the polygon as vertices, and, taking them in order on the polygon, denote them by $A_1, A_2 \cdots A_n$. By connecting the arcs of the great circles A_iO' with O by planes, which is possible since O, being on the prolonged radius $O'C$, lies in the planes of these circles, we may reduce the conical figures to a sum, with proper signs, of

conical figures $O\text{-}O'A_iA_{i+1}$. These figures have spherical bases, two plane faces, and one curved one. To reduce them, we draw in the plane of the arc A_iA_{i+1} to cut OO' at D, and also the conical surface with vertex C, and base arc A_iA_{i+1}. We then see that

$$O\text{-}O'A_iA_{i+1}=O\text{-}DA_iA_{i+1}\pm(C\text{-}O'A_iA_{i+1}-C\text{-}DA_iA_{i+1}).$$

In case A_iA_{i+1} happens to be a great circle, D coincides with C, and the last term vanishes. In general, $O\text{-}DA_iA_{i+1}$ and $C\text{-}DA_iA_{i+1}$ are cones, whose reduction we have already considered, while $C\text{-}O'A_iA_{i+1}$ is a spherical cone. It may be reduced by first decomposing its base into right spherical triangles and sectors on a sphere by the method of section 2, and then joining the lines so introduced to C by planes and conical surfaces.

We have thus reduced our volume to figures of the types mentioned in the theorem.

4. Areas. When a figure composed of planes and spheres is specified by parameters which are coördinates of points, coefficients in equations of surfaces, or lengths and angles, parameters of all these types may be calculated in terms of them for the simpler figures obtained by dissection mentioned in theorem I. The calculation will obviously introduce at most algebraic and trigonometric functions, or their inverses. But, from elementary geometry, we have as the area of a triangle of base b and altitude h, $\frac{1}{2}bh$; of a sector of radius r and angle A, $\frac{1}{2}r^2A$, of a right spherical triangle with angles A, B, $\pi/2$ on a sphere of radius r, $(A+B-\pi/2)r^2$ and of a sector on a sphere with angle A, side R, on a sphere of radius r, $Ar^2(1-\cos R)$. Hence, from theorem I and the additive character of areas, we have:

THEOREM III. *Any area in a figure composed of spheres and planes is an elementary function of the paramteers of the figure.*

5. Volumes. For volumes we note that the parameters for the figures mentioned in theorem II may be computed in terms of those of the original figure. For the simpler figures, we have, from elementary geometry, as the volume of a cone or pyramid of base B and height h, $(1/3)Bh$, and as that of a spherical cone or pyramid with base of area B on a sphere of radius r, $(1/3)Br$. In each case, B may be computed in terms of the parameters by theorem III, and, since volumes are additive, we have:

THEOREM IV. *Any volume in a figure composed of spheres and planes is an elementary function of the parameters of the figure.*

6. Centers of gravity of surfaces. Since the coördinates of the center of gravity of a composite area are easily computed in terms of the areas and coordinates of the centers of gravity of the parts, and the question of areas has been disposed of in theorem III, we need merely show that the coördinates of the center of gravity of the simple figures mentioned in theorem I are elementary

functions. For the plane figures, we readily find by integration that the center of gravity of a triangle is at the median point, while for a circular sector of angle A and radius r, it is on the radius which bisects it at a distance $\frac{4r \sin (A/2)}{3A}$ from the center.

For spherical figures, we first prove a general result.[1] If z denotes the distance to a diametral plane, S refers to the spherical surface, S_z to the projection of this surface on this diametral plane, θ the angle made by the tangent plane to the sphere and this plane at any point, or by the radius and ordinate at the point, since these are respectively normal to the two planes, and R is the radius of the sphere, we have:

$$A\bar{z} = \int\int z\, dS = \int\int (z/\cos\theta)\, dS \cos\theta = \int\int R\, dS_z = RA_z .$$

This shows that the coördinates of the center of gravity of a figure on a sphere with respect to axes through its center, and hence those with respect to any axes, may be computed in terms of the areas of the projections of this figure on the coördinate planes. The figures we are concerned with are spherical right triangles and sectors on a sphere. These will project into figures bounded by straight lines and arcs of ellipses. These figures, by the method of section 2, may be decomposed into triangles and elliptic sectors, and the areas of the latter found from those of the circular sectors of which they are the orthogonal projections by introducing the proper scale factor. As the computation we have outlined nowhere introduces anything beyond elementary functions, we have demonstrated:

THEOREM V. *The coördinates of the center of gravity of any area in a figure composed of spheres and planes is an elementary function of the parameters of the figure.*

7. Centers of gravity of volumes. From theorem IV, and the method of finding centers of gravity of composite volumes, we see that we have merely to dispose of the centers of gravity of the solids mentioned in theorem II. These are easily thrown back to the result of the preceding section. For, a cone or pyramid may be thought of as made up of elements parallel to, and similar to, the base. The centers of gravity of these elements are all on the line joining the vertex with the center of gravity of the base, and since the masses of the elements are proportional to the square of the distances from the vertex to them, measured along this line, our problem is identical with that of finding the center of gravity of a rod coincident with this line whose density varies as the

[1] *Cf.* P. Appell, *Traité de Mécanique Rationnelle*, Paris, 1919, 4th edition, problem 33, p. 171.

square of the distance from one end. By integration, we easily find that for such a rod the center of gravity is 3/4 of the length of the rod from this end.

For a spherical cone or pyramid, we divide into spherical layers concentric with, and similar to the base, and apply the preceding argument, using instead of distance from the vertex to the element, distance to its center of gravity, which is proportional to it.

Thus we find, for a cone, pyramid, spherical cone or spherical pyramid, the center of gravity is on the line joining the vertex to the center of gravity of the base, and is 3/4 the distance from the vertex to this point. In view of theorem V, this proves

THEOREM VI. *The coördinates of the center of gravity of any volume in a figure composed of spheres and planes is an elementary function of the parameters of the figure.*

8. Moments of inertia of surfaces. The moment of inertia of a composite figure about any axis is simply the sum of the moments of inertia of its several parts about that axis. This reduces our problem to the consideration of the moment of inertia of a figure of the type mentioned in theorem I about any axis. But, from the relation between the mass, distance between an axis and a parallel axis through the center of gravity, and the moments of inertia about these axes, we see that in view of theorems III and V, we may pass from the moment of inertia about any axis to that about a parallel axis through the center of gravity, and thence to any parallel axis. Thus we need only compute the moments of inertia of our standard figures about axes through a single point, for example, the origin. But for these moments of inertia, we need merely to know the integrals and products of inertia: I_x, I_y, I_z, I_{xy}, I_{yz}, I_{xz}, where $I_x=\int x^2 dm$, $I_{xy}=\int xy\, dm$, and the other quantities are defined similarly.

For the plane figures, we may compute the results directly by the ordinary methods. We find, for a right triangle with sides a and b taken as x and y axis respectively:

$$I_x=Ma^2/6\,, \qquad I_y=Mb^2/6\,, \qquad I_{xy}=Mab/12\,, \qquad I_z=I_{xz}=I_{yz}=0\,.$$

For a circular sector of radius r and angle A, taking the radius bisecting it as the x axis, and the perpendicular to it through the vertex in the plane of the sector as the y axis, we find:

$$I_x=\frac{Mr^2}{4}\left(1+\frac{\sin A}{A}\right), \qquad I_y=\frac{Mr^2}{4}\left(1-\frac{\sin A}{A}\right), \qquad I_{xy}=I_z=I_{xz}=I_{yz}=0.$$

For spherical figures, we take the origin at the center of the sphere on which our triangle or sector lies, and examine the integrals of inertia. Let R be the radius of the sphere, S refer to the spherical surface, and S_z refer to the projection on the xy plane. Let θ denote the angle made by the tangent plane to the

sphere at any point with the xy plane, *i. e.* the angle made by the radius to this point and the ordinate, respectively normal to these two planes.

We have:

$$I_z = \iint z^2 dS = \iint z(z/\cos\theta)dS\cos\theta = \iint zR dS_z = R\iint z dS_z .$$

This last integral may be interpreted as the volume bounded by A, A_z, and a cylindrical surface with elements parallel to the z axis, and computed from this interpretation. Since A is a spherical triangle, or sector on a sphere, its projection A_z is a figure bounded by three elliptic arcs, which may in particular be straight lines or circles. We draw in the chords of these arcs and planes through them parallel to the z axis. These divide our original volume into a portion whose base is a triangle, and three parts whose bases are segments of ellipses. The portion with triangular base is formed entirely of planes and spheres, and hence its volume may be computed by theorem IV. For the other pieces, we use a further dissection. Let the base of our figure (Fig. 2) be bounded by the straight line $B_z C_z$, the projection of the arc BEC, and the elliptic arc $B_z D_z C_z$, the projection of the arc BDC. D and E are arbitrary points, inserted to facilitate reference to the arcs. Since arc BDC was one of the arcs bounding A, it is a circular arc. We draw in its plane, which cuts the plane of BEC in chord BC. The volume bounded by the planes BDC and BEC and the sphere may be computed by theorem IV. The volume of the remainder of our figure may be computed by noting that, if plane BDC were parallel to the z axis, its volume would be zero, and if it is not parallel to the z axis, its equation may be solved for z and written in the form $z = ax + by + c$. Thus the volume of the figure bounded by BDC, $B_z D_z C_z$ and the plane and cylinder is:

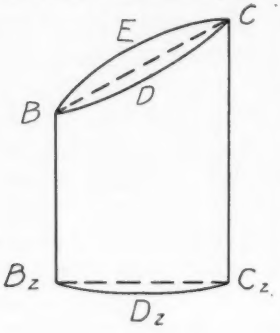

FIG. 2

$$\iint z\, dS_z = a\iint x\, dS_z + b\iint y\, dS_z + c\iint dS_z$$

$$= aA'\bar{x} + bA'\bar{y} + cA' .$$

Here A' is the area $B_z D_z C_z$, which may be computed by finding the area BDC by theorem III, and multiplying by the cosine of the inclination of its plane, or $1/\sqrt{1+a^2+b^2}$, and $\bar{x}$, $\bar{y}$ are the x and y coördinates of the center of gravity of $B_z D_z C_z$, which are equal to those of BDC, from which it is projected, and hence may be computed by theorem V. Thus I_z, and similarly I_x and I_y may be computed.

For the products of inertia, we have:

$$I_{xz} = \int\int xz\,dS = \int\int x(z/\cos\theta)\,dS\cos\theta = \int\int xR\,dS_z = RA_z\bar{x}\,.$$

But, S_z is a plane figure bounded by straight lines and arcs of ellipses. Hence its area, A_z, and the x coördinate of its center of gravity, $\bar{x}$, may be computed by the method of section **6**. The remaining products of inertia, I_{xy} and I_{yz} may be computed similarly.

Thus we have now proved

THEOREM VII. *The moment of inertia about any axis of any area in a figure composed of spheres and planes is an elementary function of the parameters of the figure.*

9. Moments of inertia of volumes. From the discussion of section 8, we see that if we compute the moments of inertia of the solids mentioned in theorem II about an arbitrary axis passing through a single point, we shall have solved the general problem. These solids are cones, pyramids, spherical cones and spherical pyramids, and the moment of inertia of such a figure about any axis through its vertex is readily thrown back to theorem VII.

For, as in section **7**, we may divide such a figure into plane or spherical layers similar to the base, with center of similitude at the vertex. Hence the radii of gyration of these sections for any given axis through the vertex will be proportional to the distances they cut off on any line through the vertex. But their masses are proportional to the squares of these distances. Thus their moments of inertia are proportional to the fourth power of these distances, and consequently, if the moment of inertia of the surface which is at the base is I_s, the moment of inertia of the volume is:

$$I = \int_0^r I_s \frac{x^4}{r^4}\,dx = \frac{rI_s}{5}\,.$$

Since x must be measured normal to the layers, r is the radius of the sphere in the case of the spherical figures, and the altitude in the case of the cones and pyramids.

We have thus shown that, for a cone or pyramid, the moment of inertia about any axis through the vertex is 1/5 the product of the altitude by the moment of inertia of the base about this axis, while for a spherical cone or spherical pyramid, it is 1/5 the product of the radius of the sphere by the moment of inertia of the base about this axis. Recalling theorems IV, VI and the relation connecting moments of inertia about parallel axes, we see that this combined with theorem VII, proves

THEOREM VIII. *The moment of inertia about any axis of any volume in a figure composed of spheres and planes is an elementary function of the parameters of the figure.*

10. Summary and further results. The essential content of the theorems we have stated is the fact that, for any surface or volume in a figure formed entirely of spheres and planes, the content, coördinates of the center of gravity and moment of inertia about any axis are elementary functions of the parameters determining the figure. Certain additional conclusions may be drawn from our proofs. Since the geometrical dissections and treatment of the fundamental integrals is capable of translation into analytic operations, we have not only shown that the final definite integrals are elementary, but that there is at least one way of handling the original multiple integrals throughout which all the intermediate indefinite integrals will be elementary.

We also note that the final results in all cases involve only algebraic, trigonometric, and inverse trigonometric functions. In fact, while we have applied our results to multiple integrals, and used a few integrals in our proofs, these are all of simple character, and could easily be replaced by the circumlocutions of elementary geometry. The spirit of our discussion is synthetic, and it would, after slight modification, become an elementary treatment of contents, centers of gravity and moments of inertia for the figures considered.

It is also to be observed that in proving our theorems we have in each case given an explicit method of computing the final result. In specific applications the process is, of course, generally considerably shorter than that which we have outlined. If the reader has any doubt of the practical value of these methods, it is suggested that he apply them to the figure formed by a sphere cut by a square prism with one diameter of the sphere as an axis, mentioned in the introduction, and try to check his results by direct multiple integration.

The class of figures to which our methods apply may be extended somewhat by orthogonal projection. For plane figures, areas are merely changed by a constant factor, centers of gravity project directly, and for a suitable choice of coördinate axes, the integrals and product of inertia, I_x, I_y, I_{xy}, are unchanged, multiplied by the square of the scale factor, and multiplied by the scale factor respectively. By first carrying out the dissection, and then projecting sectors with one elliptic arc into circular sectors, we obtain:

THEOREM IX. *For any plane figure bounded by straight lines and arcs of ellipses or circles, the area, coördinates of the center of gravity, and moment of inertia about any axis are elementary functions of the parameters determining the figure.*

We have used special cases of this theorem in sections 6 and 8.

We may use analogous considerations in three dimensional space, using simple three dimensional strains instead of orthogonal projection. Such strains change

volumes, centers of gravity, and integrals of inertia in a simple manner. We shall not attempt to characterize any general class of figures involving ellipsoids which may be transformed by simple strains into figures involving spheres, confining ourselves to figures with a single ellipsoid, which can obviously be so transformed. Consequently we formulate:

THEOREM X. *For any volume in a figure composed of a single ellipsoid and a number of planes, the volume, coördinates of the center of gravity, and moment of inertia about any axis are elementary functions of the parameters of the figure.*

Our original discussion of figures involving spheres and planes can probably be carried out for figures of more dimensions involving hyperspheres and hyperplanes. The question of finding other fairly wide classes of figures in three space for which the problems we have treated lead to elementary integrals also suggests itself.

ON THE ORIGIN OF THE TERM "ROOT"

By SOLOMON GANDZ, Rabbi Isaac Elchanan Theological Seminary, New York City

Starting from the investigation of a philological problem the writer came, with the kind help of Professor David Eugene Smith,[1] to some results which may interest students of mathematics, and which he therefore wishes to submit to the readers of this MONTHLY.

The term "root" has its origin in the Arabic. "Latin works translated from the Arabic have *radix* for a common term, while those inherited from the Roman civilization have *latus*."[2] *Radix* ("root") is the Arabic *jadhr*, while *latus* (Greek, πλευρά, *pleura*, meaning "rib" or "side") is the side of a geometric square.

It is certainly rather strange that such a term as "root" should be used in this connection. It suggests that if the basic number is a root, the square might be a bush, and so on up in a kind of a mathematical garden.

The Chinese, indeed, do use the word *kun* to mean root, grass, and shrub, and the Hindus also use the word *mûla* for the root of a plant, but this was very likely due to the Arabic influence, which is so often seen in China and which may have spread into India by way of China. It is, however, a fact that thus far we have no satisfactory explanation as to why this botanical term should have found place in the theory of numbers. Considered geometrically we can see why the line represented the basic number, and why the square represented its

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[1] Professor Smith has been most helpful in discussing with me the mathematical side of the question and giving me his advice and suggestions. In addition to this he has placed at my disposal his private library, which is especially rich in old and rare Arabic books and manuscripts, without which this paper could not have been written.

[2] See Smith, *History of Mathematics*, vol. II, p. 150.

second power, and the cube its third power. We can also see why a race of geometers should have used πλευρά, πλευρόν (*pleura, pleuron*) for the first power, and why they would not use such a term as ρίζα[1] (*riza*, "root") in this connection. It is also clear that the Latin writers, whose theoretical mathematics came wholly from the Greeks, would naturally use *latus* ("side") for the same purpose. Therefore the problem before us, as already set forth clearly by Professor Smith,[2] is to find out whether the medieval Latin authors were correct in their translation of the Arabic word *jadhr* by *radix* ("root").

The writer's doubts about the matter were aroused by the fact that the Hebrew scholars of the earlier Middle Ages did not commonly translate the word *jadhr* by *shoresh* ("root") but by *gader* which means "wall," "side," "limit," or "boundary."[3] For example, Abraham bar Chiia (*c.* 1100), commonly known as Savasorda, in his Hebrew geometry, never uses *shoresh* ("root") but always *gader*. Ibn Ezra seems to be the first who introduced into Hebrew, in his *Sefer ha-Mispar* ("Book of Arithmetics") the term "root," using both the traditional word *gader* and the later form *shoresh*. Even in the first line of his *Sefer ha-Echad* he says *shoresh wiyesod* ("root and basis"), showing that he knew the Arabic *jadhr*, was doubtful how to translate it, and thought it best to give the two meanings, *shoresh* ("root") and *yesod* ("basis"). In his Latin translation of Savasorda's work (*Liber Embadorum*, ed. Curtze, Leipzig, 1902). Plato of Tivoli, in his early definitions (p. 18, §§ 19, 20) uses *latus* for *gader*, but later (p. 32, § 8; p. 34, §§ 9, 10; p. 91, § 20; p. 92, §§ 23, 24; p. 98, 4) he uses *radix* instead. The corresponding Hebrew text, *Hibbûr ha-Meshiḥah we ha Tisboreth* (Berlin, 1912), always uses the word *gader* (p. 25, § 46; pp. 26, 27, §§ 46, 47; p. 51, §§ 91, 93, 93a, *et passim*).

These doubts as to the correct translation from the Arabic were strengthened by the statement of Professor Smith that the idea of "root" had its origin solely in the Arabic writings. The writer therefore started out to investigate the real meaning of the word *jadhr*, not depending upon lexicons or the Latin versions of the thirteenth century, but seeking the meaning as it appears in the manuscripts of the old Arabic mathematicians themselves.

Mohammed ibn Mûsâ al-Khowârizmî[4] (*c.* 825), is the oldest Arabic mathematician of much prominence, and it was his *'Ilm al-jabr wa'l muqabalah* that gave to Europe both the name and the foundation principles of algebra. His

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[1] See also Tropfke, *Geschichte der Elementar-Mathematik*, vol. I, p. 214, note 864, and Ruska, *Zur, arabischen Algebra*, p. 68, note 1.

[2] *History*, II, 150.

[3] See Rosin, *Magazin für die Wissenschaft des Judentums*, V, pp. 47 *seq.*, and Steinschneider, *Ibn Ezra* pp. 94 and 110; reprinted also in his *Gesammelte Schriften*, (ed. Marx-Malter), p. 475.

[4] Smith, *History*, I, 170.

chapter *Bâb al Misâcha* ("on geometry")[1] begins as follows: "Know that the meaning of the expression 'one in one' is an 'area'; and its meaning is one cubit (in length) in one cubit (in breadth). And every roof[2] of equal sides and angles which has one cubit for each side is a (square) unit. But if it has two cubits for every side and has equal sides and angles, then the whole roof is four times the roof which has one cubit in one cubit. . . . . And in every square roof of equal sides one of its sides (multiplied) in a square unit is its *jadhr*; or if the same be multiplied in two (square units) then it is like two of its *jadhrs*, whether this roof be small or great." Here we have a clear definition of the *jadhr*. It is a side multiplied with a square unit. *Jadhr* not only means "root" but also[3] "basis," "foundation," "lowest part." Mohammed ibn Mûsâ was a clear thinker. He knew that we can never get a square measure by multiplying one side by one side, in the ordinary meaning of multiplication, since the side has only one dimension. Therefore he begins his chapter on areas by introducing the new notion of a square unit. Then he says that in order to get the area of any figure we must first multiply the one side by the square unit; this is then the basis to be multiplied by the other side. We do not multiply one side by one side, but one *jadhr*, representing the square basis, by one side representing the number. The same definition and idea is also to be found elsewhere in his algebra.[4] It is also found in the old Hebrew geometry, the *Mishnath ha-Middoth* ("Theory of Measures"),[5] as follows: "What is a quadrangle equal as to sides and angles? Ten from every side. Multiply length on breadth, and the product is the area, namely hundred. The one side is its one *'iqqar*, its two sides are its two *'iqqars*, and so three and four." Now this last sentence has no sense as it stands. But *'iqqar* is just the Hebrew word for *jadhr*, and its meaning is "root," "basis," or "lowest part." Considering the fact that there are striking parallels between *Mishnath ha-Middoth* and al Khowârizmî, and that the latter was

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[1] See the Rosen edition, London, 1831; Arabic text, p. 50 *seq.*; English text, p. 70 *seq.* Rosen's translation is faulty in certain places and it has been corrected by the present writer. In the study of the Arabic text he has been greatly helped by having access to the manuscript in Professor Smith's library described in this MONTHLY for 1925, p. 395.

[2] This is the original meaning of *satch*, later used for surface. It corresponds to the Hebrew *gag* (Greek στέγη, *stegé*, "roof," "surface"); see *Mishnath ha—Middoth*, ed. Hermann Schapira, p. 14, note 3.

[3] See Lane, *Arabic-English Dictionary*.

[4] Arabic text, p. 11, line 11; and still more clearly in p. 13, line 9 seq., where it reads: "and one of its sides multiplied into the (square) unit is its *jadhr* . . . . and we make the other side, namely *hz* (three), and this is the number of its *jadhrs*."

[5] The author is not known, but he may have lived even as early as the beginning of the Christian era. See Smith, *History*, I, 174; M. Steinschneider, *Festschrift Zunz* (Berlin, 1864); H. Shapiro, *Abhandlungen zur Geschichte der Mathematik*, III, 3.

possibly influenced by his Hebrew predecessors,[1] we may venture to correct the quoted passage in his algebra to read as follows: "And its one side into one (square unit) is its one *'iqqar*, and into two (square units) is its two *'iqqars*," and so on.

These passages in the *Mishnath ha-Middoth* and al-Khowârizmî have always puzzled historians. Rosen (p. 71, lines 17 and 19), Aristide Marre in his French translation (Rome, 1866, quoted by Shapiro, p. 25, note 3), Shapiro himself (*loc. cit.*), and Ruska (*loc. cit.*, pp. 104–105) fail to give any explanation of them, and Ruska confesses frankly that he can find no meaning in them.

The explanation here given, however, is perfectly clear, and is based upon the meaning of *jadhr* and *'iqqar* in two of the earliest Arabic and Hebrew mathematical texts in which the terms appear. The term does not mean "root," but "square basis," that by whose multiplication we get the square area. This was the reason why *jadhr* was used by later writers, such as Omar Khayyam, as the basic number of a square number. The latter did not know anything more about the original meaning of *jadhr*, and he used the word[2] loosely for *ḍil'* (Greek, πλευρά), which means "rib" or "side."[3] But he,[4] like al-Khowârizmî, still knew that it was a concrete number in opposition to an abstract number.

By the time of Beha Eddîn (*c.* 1600) the original meaning was entirely forgotten. In his *Kholâsat al-Ḥisâb* ("Essence of Arithmetic")[5] he says: "What is multiplied into itself is called *jadhr* in arithmetic, *dil'* ("side") in geometry, and *shai'* ("thing," "cause") in algebra." He certainly understood by *jadhr* the abstract "root." In this misunderstood form the term *jadhr* found its way into medieval Latin as *radix*.

Professor Smith is right in calling attention to the different sources of the two terms *latus* and *radix*. The fact suggests the two leading paths by which ancient science found its way into medieval Europe. The one led from Greek to Latin and thence to other western languages. This route has been quite thoroughly explored. The other road may be characterized as the Greek-Arabic-Latin-modern-language one, and the study of this pathway has only just begun.

Many other terms than *radix* may be based upon an imperfect understanding of the Arabic sources. On the other hand thousands and thousands of Arabic

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[1] Perhaps even by this work, since the date is so uncertain, or perhaps the two were influenced by some common source which is now unknown. Such a state of affairs is found in the Latin translations from the Arabic versions of Euclid. See the argument of Shapiro in the preface to his edition.

[2] See the Arabic text, p. 5, lines 9, 10, and at the bottom. After p. 20 he uses only *ḍil'*.

[3] Arabic text, p. 4, lines 3, 8. [4] Arabic text, p. 3, line 10.

[5] Nesselmann ed., Berlin, 1843; Arabic text, p. 15; German transl., p. 14. There are two Arabic manuscripts of this important work in Professor Smith's library, and these the author has consulted in the preparation of this article.

manuscripts on mathematics, astronomy, medicine, and the other sciences are still reposing unread by modern Arabists, in the great libraries of Europe. The fact that a number of very important ones are in Professor Smith's library goes to show that abundant source material is likely to be found even in the New World,—for example, in the remarkable collection now in the Jewish Theological Seminary in New York. A thorough study of the scientific manuscripts in Arabic is of the greatest importance for the history of human knowledge and of the foundations of our modern civilization. Our history should be no longer a history of wars and massacres but a history of the progress of the human spirit, of human culture, of human rights, and of human liberties. The present interest in the history of science is evidence that this new conception of the story of the race is beginning to be realized.

QUESTIONS AND DISCUSSIONS

EDITED BY TOMLINSON FORT, Hunter College, Park Ave. & 68th St., New York, N. Y., and H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

REPLIES TO QUESTIONS

46 [*1922*, 210]. A geometrical construction will often become impracticable in special cases where some of the construction points are imaginary, although the final result is real. Is there any system according to which a construction failing in this way may be replaced by one that will work?

For example, let a line l_1 cut a circle c_1 in P_1 and Q_1, and let a line l_2 cut a circle c_2 in P_2 and Q_2. If the four points are real, the intersection of P_1P_2 with Q_1Q_2 and of P_1Q_2 with P_2Q_1 can be found directly. If the four points are imaginary, the intersections will still be real, and there ought to be a simple way of getting them as the points common to a line and a circle.

I. REPLY BY P. H. DAUS, University of California, Southern Branch.

Problems of the type suggested here are essentially solved in several texts on projective geometry. The interested reader will find such a discussion in chapter XII of Dowling's *Projective Geometry*. In view of the special problem proposed it is interesting to develop the ideas entirely from the standpoint of circles. Proofs of the results stated below will be found in texts on college geometry, for example, Durell's *Plane Geometry for Advanced Students*, chapters X and XIII.

If A and B are two points such that the tangents from them to two circles are equal, then the line AB contains all such points and is called the radical axis of the circles. All circles which have the same radical axis have their centers on a line perpendicular to the radical axis, and are said to be coaxal. Coaxal systems may be of two types. A system of the first type is such that all the circles have two real points L_1 and L_2 in common; and that of the second, is such

that the circles have no real points in common. In this case there are two point circles belonging to the system, known as the limiting points L_1, L_2. This system is characterized by the fact that every circle of the system is orthogonal to the circle on L_1L_2 as diameter. Associated with a system of either type is a system of the other type, such that the limiting points of one are the common points of the other. Every circle of one system is orthogonal to every circle of the other system.

A system of coaxal circles determines on its axis an involution of points. In the system of the second type, pairs of corresponding points are harmonic conjugates with respect to the limiting points, for $OA_1 \cdot OA_2 = OL_1^2$. L_1 and L_2 correspond to themselves and are called the double points of the involution. In the system of the intersecting type $OP_1 \cdot OP_2 = -OL_1^2$ and consequently the double points are conjugate imaginaries. The involution on the line P_1P_2 (called an elliptic involution) is uniquely determined by any circle of the associated system and the imaginary double points are common to every circle of the associated system. This provides us with the means for the solution of the problem in question.

Before proceeding, however, we shall need a few more statements. In an elliptic involution there is one and only one pair of corresponding points, which are harmonic conjugates with respect to a given pair P_1P_2. They are the intersections of the line P_1P_2 with the circle of the system orthogonal to the circle on P_1P_2 as diameter. If the points of an elliptic involution be joined to a real point we have at that point an involution of rays, which defines a pair of conjugate imaginary lines, joining the point to the pair of conjugate imaginary points.

CONSTRUCTION: Let the two given lines meet at A. Let C_1 be the center of the first circle, and draw C_1O_1 perpendicular to l_1. Through A draw the circle orthogonal to c_1, and whose center is on l_1. Let it cut l_1 again in A_1 and C_1O_1 in L_1 and L_1'. Through L_1 and L_1', draw the circle orthogonal to the circle AA_1, determining on l_1 the points B, B_1, such that ABA_1B_1 are four harmonic points. Similarly determine the harmonic range $AB'A_1'B_1'$ on l_2. Then the lines A_1A_1', BB', B_1B_1' will meet in a point P, and the lines A_1A_1', BB_1', B_1B' in Q. P and Q are the desired points, for the involution at Q defines a pair of imaginary lines through Q to ABA_1B_1 as well as to $AB_1'A_1'B'$, or Q is the intersection of the imaginary lines joining ABA_1B_1 to $AB_1'A_1'B'$. Similarly P is the intersection of the imaginary lines joining ABA_1B_1 to $AB'A_1'B_1'$.

II. REPLY[1] BY NATHAN ALTSHILLER-COURT, University of Oklahoma.

The issue of the MONTHLY in which this question appears happens, by some queer coincidence, to contain (*1925*, 486-492) the bibliographical references to

[1] The notation has been changed so that the second paragraph of the problem reads as follows: For example, let a line s cut a circle (C) in P and Q and let a line s' cut a circle (C') in P' and Q'. If

the original sources where the geometric interpretation of imaginary quantities has been elaborated. This answers the first half of the question. For the benefit of those who would prefer to consult a more accessible work I may mention J. L. S. Hatton's *The Theory of the Imaginary in Geometry*, Cambridge University Press, 1920.

Let us now consider the particular problem proposed. Before we can attempt to solve the problem, we must first make clear what we want to accomplish. When we say that two real points X and X' can be found such that $X \equiv (PP', QQ')$, $X' \equiv (PQ', P'Q)$ when the points P, P', Q, Q' are imaginary, we simply state the analytical fact that the two pairs of linear equations with complex coefficients which represent the two pairs of lines considered, have each a set of real solutions, and these solutions may be interpreted as two points in the plane of the figure. But there is nothing in this statement upon which to base a geometric solution of the problem. However the fact that two real points X, X' may be found whether the points P, P', Q, Q' are real or imaginary, makes us suspect that the points X, X' must bear a certain geometric relation to the given figure, a relation which subsists whether the lines s, s' meet or do not meet the circles (C), (C'). If such a property can be discovered, we shall have a basis for a geometric solution of our problem. The solution is subject to the restriction that when the points of intersection of the given lines with the circles become real, the points X, X' thus determined must coincide with the points X, X' determined by the direct method.

Suppose then, for the present, that the points P, P', Q, Q' are real. Let M, N, be a pair of points on s harmonically separated by P, Q. The lines XM XN which join X to the points M, N will meet the line s' in two points, say, M', N' which are harmonically separated by P', Q'. Vice versa, if we take on s' a pair of points M', N' separated harmonically by P', Q', the lines XM', XN' will meet s in two points, say, M, N, harmonically separated by P, Q. Similarly for the point X'.

The points M, N are conjugate with respect to the circle (C) and are a couple of the involution of conjugate points with respect to (C) of which s is the base, and P, Q the double points. Similarly for the points M', N' of s'. It is thus seen that the point X (or X') is the center of an involution of rays which is cut by each of the two given lines in pairs of points conjugate with respect to the given circles, or, in other words, that X (or X') is the center of an involution of rays which is perspective to both involutions of conjugate points on the two given lines. It may readily be proved that the points X, X' are the only points in the plane having this property.

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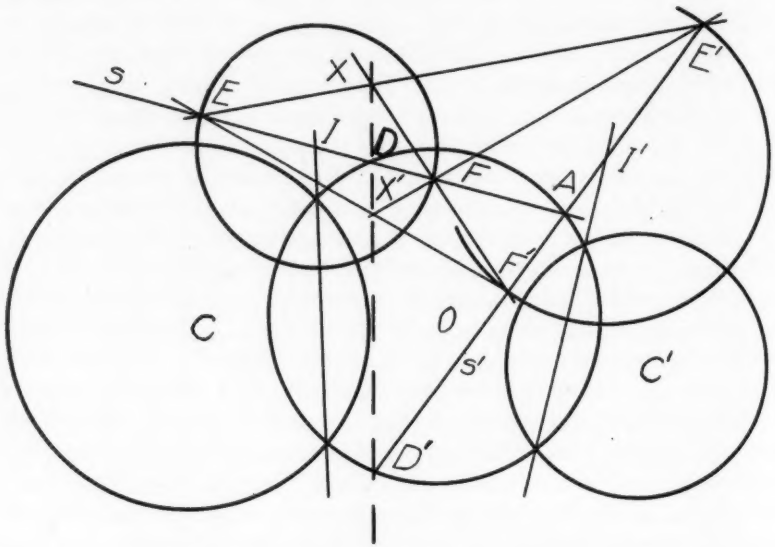
the points P, Q, P', Q' are real the intersections of PP' with QQ' and PQ' with $P'Q$ may be found directly. If the four points are imaginary, the intersections named will still be real, and there ought to be a simple way of getting them as the points common to a line and a circle.

Now when the lines s, s' do not meet the circles (C), (C') respectively, they are still the bearers of involutions of conjugate points with respect to the circles, these involutions being elliptic in this case, while they were hyperbolic when the points P, P', Q, Q' were real. To find the points $X \equiv (PP', QQ')$, $X' \equiv (PQ', P'Q)$ when the lines s, s' do not meet the respective circles may therefore mean to find the points which are the centers of involutions of rays each perspective to the two elliptic involutions of points on s and s' simultaneously. Two and only two such points may, in general, be found. The construction follows.



Through the point of intersection A of the given lines s, s' draw a circle (O) orthogonal to the given circles (C), (C') meeting s again in the point D, and s' in D'.[1] Let I be the point of intersection of the common chord of (O) and (C) with the line s, and I' the point of intersection of the common chord of (O) and (C') with the line s'. With I as center draw a circle orthogonal to (C) meeting s in the points E, F. Let E', F' be the analogous points on s'. The two pairs of lines EE', FF'; EF', $E'F$ determine the two required points X, X'.

To justify this construction we may observe that the two pairs of points A, D; E, F are conjugate with respect to (C), hence these two pairs of points determine the involution of conjugate points on s with respect to (C). Moreover

[1] For the details of this and the following construction see, for instance, Nathan Altshiller-Court, *College Geometry*, Johnson Publishing Company, Richmond, Va., 1925, Chap. VI.

the four points $ADEF$ are harmonic. Similarly for the points A, D', E', F'. The two harmonic ranges having the point A in common, the lines EE', FF' intersect on the line DD', so that the pencil of lines $X(ADEF)$ is perspective both to the involution of conjugate points on s with respect to (C), and to the involution of conjugate points on s' with respect to (C'). Hence any pair of lines projecting from X a pair of conjugate points of the involution on s will meet s' in a pair of conjugate points with respect to (C'). Similarly for X'.

DISCUSSIONS

I. A Curious Algebraic Function.

By Frank Schlesinger, Yale University Observatory

In the discussion of measures (x) of the positions of star images on certain photographic plates that extend from $-X$ to $+X$ an empirical term of the form Px^n was introduced and a least-squares solution was carried out to secure the best agreement between the measured positions and those known in advance, n being assumed and P being the unknown. The star images are uniformly distributed on the plate. The question arose as to the degree of completeness with which Px^n represents the empirical term if its true form is Kx^{n+2}. In that case the least-squares solution has made a minimum the sum of the squares of the quantities

$$Kx^{n+2}-Px^n .$$

If we square, integrate with respect to x from $-X$ to $+X$, differentiate with respect to P and set the derivative equal to zero we obtain

$$P=K\frac{2n+1}{2n+3}X^2$$

and the difference between the two terms is

$$Kx^{n+2}-K\frac{2n+1}{2n+3}X^2x^n .$$

This has a (numerical) maximum value at

$$x=X\left(\frac{n}{n+2}\cdot\frac{2n+1}{2n+3}\right)^{\frac{1}{2}}.$$

At the edge of the plate $(x=\pm X)$ occurs another maximum, of course not in the algebraic sense. The ratio of the two maxima is

$$\left(\frac{2n+1}{n+2}\right)^{(n+2)/2}\left(\frac{n}{2n+3}\right)^{n/2}.$$

If we call this function $f(n)$ and study it for values of $n \geqq 1$, we observe the following curious property: $f(1)=0.44721$, $f(2)=0.44643$, $f(3)=0.44631$, $f(4)=0.446281$, $f(5)=0.446270$, $f(6)=0.446266$, $f(\infty)=0.44626032=2e^{-3/2}$.

We see then that, for all values of n (integral or fractional) not less than unity, this function has the same value within less than 0.001; the value for $n=2$ differs from the limit by only one part in 2600.

II. Quadrangles with the Brocard Property

By P. S. Wagner, Johns Hopkins University

The Brocard points and angle of a triangle, and also of special quadrangles, have been taken up and discussed in detail.[1] It is the purpose of this paper to take up this question for the general quadrangle.

If the convex quadrangle a, b, c, d is such that the angles formed by joining an internal point x to the vertices, *i.e.*, xab, xbc, xcd, xda, are all equal to the same angle θ, and if we denote the lines xa, xb, xc, xd by r_1, r_2, r_3, r_4, respectively, and the angles dxa, axb, bxc, cxd by $\alpha, \beta, \gamma, \delta$, respectively, then

$$\frac{r_1}{r_2}=\frac{\sin(\beta+\theta)}{\sin\theta}; \qquad \frac{r_2}{r_3}=\frac{\sin(\gamma+\theta)}{\sin\theta}$$
$$\frac{r_3}{r_4}=\frac{\sin(\delta+\theta)}{\sin\theta}; \qquad \frac{r_4}{r_1}=\frac{\sin(\alpha+\theta)}{\sin\theta} \tag{1}$$

from which, by multiplication,

$$(\sin\alpha\cot\theta+\cos\alpha)(\sin\beta\cot\theta+\cos\beta)(\sin\gamma\cot\theta+\cos\gamma)(\sin\delta\cot\theta+\cos\delta)=1. \tag{2}$$

If we let $\cot\theta=\pm i$, we see that the equation is satisfied. Hence the quartic (2), which is

$$(\cot\theta+\cot\alpha)(\cot\theta+\cot\beta)(\cot\theta+\cot\gamma)(\cot\theta+\cot\delta)=\csc\alpha\csc\beta\csc\gamma\csc\delta, \tag{3}$$

is identically

$$(\cot^2\theta+1)(\cot^2\theta+A\cot\theta+B)=0 \tag{4}$$

so that, if we write (3) as

$$\cot^4\theta+S_1\cot^3\theta+S_2\cot^2\theta+S_3\cot\theta+S_4\csc\alpha\csc\beta\csc\gamma\csc\delta=0 \tag{5}$$

we have,

$$A=S_1 : B+1=S_2 : A=S_3 : B=S_4\csc\alpha\csc\beta\csc\gamma\csc\delta .$$

Hence the general condition on the angles of the quadrangle is that

$$1-S_2+S_4=\csc\alpha\csc\beta\csc\gamma\csc\delta \tag{6}$$

and since now

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[1] See *e. g.* Casey: *A Sequel to Euclid*, 6th ed., p. 172; Casey: *A Treatise on the Analytic Geometry of the Point, Line, Circle and Conic Sections*, p. 64; Coolidge: *A Treatise on the Circle and the Sphere*, p. 61.

$$\cot^2\theta + A\cot\theta + B = 0 \text{ becomes} \tag{7}$$

$$\cot^2\theta + S_1\cot\theta + S_2 - 1 = 0 \tag{8}$$

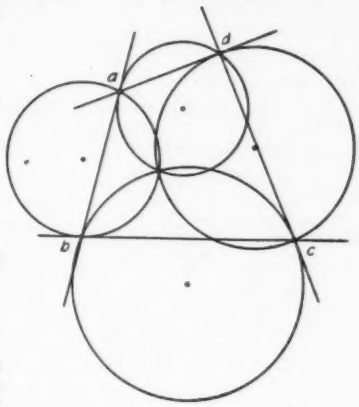
we have $\cot\theta$ in terms of the cotangents of the angles of the quadrangle.

If a, b, c, d are the complex names of the vertices of our quadrangle, we have

$$\frac{x-a}{b-a} = \rho t : \frac{\bar{x}-\bar{a}}{\bar{b}-\bar{a}} = \rho/t \tag{9}$$

so that

$$\begin{aligned}
(x-a)(\bar{a}-\bar{b}) &= t^2(a-b)(\bar{x}-\bar{a}) \\
(x-b)(\bar{b}-\bar{c}) &= t^2(b-c)(\bar{x}-\bar{b}) \\
(x-c)(\bar{c}-\bar{d}) &= t^2(c-d)(\bar{x}-\bar{c}) \\
(x-d)(\bar{d}-\bar{a}) &= t^2(d-a)(\bar{x}-\bar{d})\,.
\end{aligned} \tag{10}$$



The condition that these four lines pass through the same point x is the vanishing of

$$\begin{vmatrix}
a(\bar{a}-\bar{b}), & \bar{a}(a-b), & (a-b), & (\bar{a}-\bar{b}) \\
b(\bar{b}-\bar{c}), & \bar{b}(b-c), & (b-c), & (\bar{b}-\bar{c}) \\
c(\bar{c}-\bar{d}), & \bar{c}(c-d), & (c-d), & (\bar{c}-\bar{d}) \\
d(\bar{d}-\bar{a}), & \bar{d}(d-a), & (d-a), & (\bar{d}-\bar{a})
\end{vmatrix}. \tag{11}$$

This, then, is the general condition on the quadrangle *abcd* that it shall have at least one Brocard Point. However, the symmetry of the determinant shows that if there is one there must be two of these points; the other one being derived from the negative sense, *i.e. adcb*, of the quadrangle. In each case, we have the same Brocard angle.

It is also of interest to note that if we impose the condition that the quadrangle shall be cyclic we may replace $\bar{a}$ by $1/a$, etc., in (11). This general condition then reduces to the condition that the four vertices a, b, c, d are harmonically separated, *i.e.* the quadrangle is Casey's[1] harmonic quadrangle

The method of locating these points geometrically, when they exist, is a direct extension of the one used by Coolidge[1] for the case of the triangle. Name the vertices of the quadrangle, in positive order, consecutively a, b, c, d, beginning arbitrarily. Then draw a circle through a touching bc at b, a circle through b touching cd at c, a circle through c touching da at d, and a circle through d touching ab at a. (Figure.) These four circles will all meet in a point which is

[1] *Loc. cit. ante.*

[1] *Loc. cit. ante.*

one of the Brocard points if there is one. The proof of this is analogous to that for the triangle. To locate the other Brocard point start at any vertex, name the consecutive vertices a, b, c, d in the negative or clockwise order and repeat the above constructions.

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS

Mathematics of Finance. By LLOYD SMAIL. New York, McGraw-Hill Book Company, 1925. 310 pages. Price $3.50.

During the past few years several books on the subject of mathematics of finance have been written. In the present volume Professor Smail has attacked the subject more vigorously than have most writers.

The work is intended as a practical text book for students in colleges of business administration with the minimum requirements in algebra. The author has added a number of sections in small print in which he has discussed the subject in a more theoretical manner. For students interested in the theory of the subject chapters XIV and XV should furnish sufficient material to carry on the work. In chapter XIV the general theory of interest and discounts is discussed with the aid of the calculus and higher analysis. In chapter XV the theory of varying annuities is presented by means of finite differences.

The book is divided into three parts; Part I deals with simple and compound interest, discounts, equation of payments, annuities, depreciation, valuation of bonds, building and loan associations.

Part II discusses the general theory of probability, the mortality table, life annuities, and life insurance. However, after such an excellent presentation of the subject in part I, one wonders why the author did not go more into detail in part II. A few more solutions of problems and illustrations would have been in keeping with the former section.

Part III gives a brief discussion of logarithms, progressions, and the binomial theorem.

Throughout the book we have a clear, concise treatment of the subject. Numerous applications and illustrations have been given and a variety of problems follow each chapter. Chapter IX is given to a summary of the most important formulas and an analysis of problems in simple and compound interest, discounts, and annuities. A general summary of all formulas is included with the tables. The tables have been chosen with great care and should answer any purpose for which they are needed.

The arrangement of the material is logical and is adaptable to any ordinary requirement of the student or instructor. On the whole this should be an acceptable text for colleges of business administration.

A. W. RICHESON.

ARTICLES IN CURRENT PERIODICALS.

Bulletin of the American Mathematical Society, volume 31, nos. 9–10, November-December, 1925; "On the Number of Elements of a Group which have a Power in a given Conjugate Set" by Louis Weisner, 492–496; "Note on the Construction of Tables of Linear Forms" by D. N. Lehmer, 497–498; "Criteria that any Number of Real Points in n-Space shall lie in an $(n-k)$-Space" by H. S. Uhler, 499–502; "Characteristic Parameter Values for an Integral Equation" by W. A. Hurwitz, 503–508; "On the Polynomial of the Best Approximation to a Given Continuous Function" by J. Shohat, 509–514; "Resolvent Sextics of Quintic Equations" by L. E. Dickson, 515–522; "Improper Double Integrals" by C. A. Shook, 524–530; "The Fundamental Region for a Fuchsian Function" by L. R. Ford, 531–539; "Some Phases of Descriptive Geometry" by W. H. Roever, 540–550.

Journal of Mathematics and Physics (Massachusetts Institute of Technology), volume 5, no. 1, December, 1925: "The Elementary Theory of Almost Periodic Functions of Two Variables" by Philip Franklin, 40–54; "Adjoint and Inverse Determinants and Matrices" by L. H. Rice, 55–64.

Proceedings of the Edinburgh Mathematical Society, volume 43, part 1, May, 1925: "Some Axi-Symmetric Determinants with Integers for Elements" by J. J. Nassau, 64–69. Part 2, November, 1925: "On the Rearrangement of Terms in a Complex Series" by S. Beatty, 85–91.

Science, new series, volume 63, no. 1625, February 19, 1926; "The Earliest Known American Arithmetic" by Louis C. Karpinski, 193–195.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB TOPICS

I. LA COURBE DU DIABLE.

By B. H. BROWN, Dartmouth College.

Ever since the middle of the eighteenth century[1] the equation

$$y^4 - x^4 - 96a^2y^2 + 100a^2x^2 = 0 \qquad (1)$$

has been extensively employed to test the abilities of students in curve tracing. The origin of the name *courbe du diable* we do not know, but in all probability the curve was summarily christened by some exasperated youth who felt strongly and expressed himself thus forcibly on the subject. Frost[2] in his excellent book says that this equation "deserves to be considered carefully as to the points which we have at present discussed," (symmetry, location, singular

[1] Cramer, *Introduction à l'Analyse des lignes courbes*, 1750, p. 19.

[2] Frost, *Curve Tracing*, 4th ed., London, 1918, pp. 26, 27, plate II, fig. 15.

points and tangents thereat, asymptotes, etc.). Lacroix,[3] Laurent,[4] Moigno,[5] Briot and Bouquet,[6] Niewenglowski,[7] and others have included this problem in their treatises. Loria[8] merely mentions it as a worthwhile example, but Teixeira[9] gives a detailed discussion (with a very inaccurate figure).

The actual tracing of the curve requires no greater ability than the Continental teachers have always expected of their students. The slope in the form

$$y' = \frac{x(x^2-50a^2)}{y(y^2-48a^2)},$$

gives a good deal of information. The polar form

$$\rho^2 = 96a^2 + 4a^2\frac{\cos^2\theta}{\cos 2\theta}$$

gives more. With the information thus gleaned we can certainly trace the curve better than Teixeira does.

But it is a little surprising that none of the authors to whose works we have access notice that (1) is a very slight variation from the degenerate locus

$$(y+x)(y-x)(y^2+x^2-100a^2) = 0 . \qquad (2)$$

We see easily that $(\pm\sqrt{50}a, \pm\sqrt{50}a)$ are singular points of (2) but not of (1); and we may now discuss how the branches of our devil-curve have broken away from each other at these points. This point of view furnishes us an excellent idea of the shape of the curve.

II. 1926 as a Centennial Year in the History of Mathematics

By W. C. Eells, Whitman College, Walla Walla, Washington

Last year in this Monthly (*1925*, 258) a list of important events in the history of mathematics for which the year 1925 was a centennial year was published. It was suggested that program committees of college and high school mathematics clubs, teachers of courses in the history of mathematics, and others might find these events of interest as suggestive of timely topics for programs and special investigations.

Below is given a similar list of events which occurred, according to one or more of the standard historians of mathematics, a whole number of centuries

[3] *Traité élémentaire de calcul différentiel et intégral*, 1837, p. 158.
[4] *Traité d'Analyse*, vol. 2, p. 185.
[5] *Leçons de calcul différentiel et de calcul intégral*, etc., vol. 1, Paris, 1840, p. 222.
[6] *Géométrie analytique*, 2nd ed., p. 197.
[7] *Cours de géométrie analytique*, vol. 2, Paris, 1895, p. 73.
[8] *Ebene Kurven*, vol. 1, p. 101.
[9] *Traité des courbes spéciales*, Fr. ed. 1908, vol. 1, p. 296.

preceding 1926, accompanied in some cases by suggestions of closely related topics.

B.C.

274. Approximate date of birth of Eratosthenes. Geography and geodesy of the ancients; prime numbers and "sieve" of Eratosthenes; the Julian calendar.

A.D.

526. Death of Boethius: Medieval geometry and arithmetic; mathematics among the Romans.

626. Wang Hs'iao-T'ung ordered to conduct examinations on the calendar. Solution of numerical cubic equations in China. (See Mikami: *The Development of Mathematics in China and Japan*, Leipzig, 1913. pp.53–56.)

826. Birth of Tabit ibn Qorra. Development of mathematics at Bagdad; Arabian translations of Euclid; magic squares.

1526. Death of Ferro. European solution of cubic equations.

1526. Introduction of present symbol for square root by Rudolph. Development of algebraic symbolism.

1626. Death of Cataldi. Continued fractions.

1626. Death of Francis Bacon. Experimental foundation of applied mathematics.

1626. Death of Gunter. Origin of trigonometric functions (cosine and cotangent); computation of tables of logarithmic functions; "Gunter's chain" and land surveying.

1626. Death of Snell. Computation of π; polar triangles; "loxodrome" and navigation.

1626. Publication of Girard's treatise on trigonometry, containing first use of abbreviations: sin, tan, and sec.

1726. Death of Nicolaus Bernoulli. The Bernoulli family; mathematics at St. Petersburg.

1726. Establishment of Hollis professorship of mathematics at Harvard College, first endowed professorship of mathematics in America. Early mathematics at Harvard and other colonial universities.

1826. Publication of Davies' *Descriptive Geometry*. Mathematics at West Point.

1826. Publication of Warren Colburn's *Arithmetic Upon the Inductive Method of Instruction*. Colburn's influence on American elementary mathematics.

1826. First publication of the *Journal für die reine und angewandte Mathematik*, or "Crelle's Journal." Mathematical periodicals.

1826. Birth of H. J. S. Smith. Development of mathematics at Oxford; number theory. (For biography see Macfarlane's *Ten British Mathematicians*.)

1826. Birth of Riemann. Riemann surfaces; elliptic non-Euclidean geometry.

1826. Publication of Abel's memoir on "Abelian Functions."

1826. Presentation of Lobachevsky's first paper on foundations of geometry at University of Kasan. Hyperbolic non-Euclidean geometry.

CLUB ACTIVITIES

THE MATHEMATICS CLUB OF BUCKNELL UNIVERSITY, Lewisburg, Pa.

[*1924*, 203]

Meetings held during the year 1924–1925 were as follows:

October 13, 1924. "As the crow flies," short cuts in arithmetic, by Professor H. S. Everett.

January 19, 1925. "Number systems" by Professor J. S. Gold.

April 20. "Some moments and I" by Professor C. A. Lindemann.

June 1. Social meeting at the home of Professor Gold.

(Report by Professor Everett)

PI MU EPSILON, Bucknell University, Lewisburg, Pa.

The Bucknell University chapter of Pi Mu Epsilon, a national scholarship fraternity for students of mathematics, was founded March 5, 1925. The following officers were elected for the remainder of the year:

Director.....Professor H. S. Everett
Vice-director.....M. F. Decker '25
Secretary.....Hulda J. Baxter '25
Treasurer.....W. I. Miller '26
Librarian.....Catherine S. Baxter '25.

Meetings held during the remainder of the year were as follows:

March 18, 1925. Business meeting.

April 28. Business meeting.

May 6. Initiation and banquet at the Cameron House.

May 8. Lecture "The total solar eclipse of 1925" by Professor J. A. Miller of Swarthmore College, illustrated by slides and a Pathé reel.

May 26. Lawn party at the home of Professor Everett.

(Report by Professor Everett)

THE MATHEMATICS CLUB OF BROWN UNIVERSITY, Providence, R. I.

[*1925*, 202]

The Committee on Program of the Mathematics Club of Brown University for the year 1925-1926 consists of Professor R. E. Gilman, Professor C. R. Adams, Winifred F. Pine '26, and Agnes A. Duffy '27. The Committee on Arrangements consists of Mr. F. C. Jonah, Mary V. Kenny '26, Hazel M. Gilbert '27, Leslie T. Fagan '26, Harvey J. Ollsen '28.

The following program is announced:

November 10, 1925. "Women as mathematicians" by Edith S. Remington '26. "Mathematica Gothica" by Joseph A. Yates '27. "The cell of the honey bee" by B. W. Shaw '28.

December 1. "Fermat" by Mildred V. Mott '27. "Graphic solution of cubic and biquadratic equations" by Jacob Goodman '26.

January 12, 1926. "Aspects of analysis situs" by Professor Marston Morse.

February 23. "Archimedes" by Nellie C. Morton '27. "Descartes Geometry of 1637" by Harvey J. Ollsen '28. "Geometrical constructions with compasses alone" by Winifred F. Pine '26.

March 23. "Too little mathematics and too much" by Professor E. B. Wilson, Harvard School of Public Health.

April 27. "History of arithmetic" by Lucy G. Russell '26. "Sun dials" by Arthur R. Tebbutt '27.

May. Picnic.

(Report by Mr. Sauté)

The Ibis Club, Women's College of Delaware, Newark, Delaware.

The officers for the year 1924–1925 were:

President.....Mary L. Marvel '26
Vice-president.....Virginia Chapman '26
Secretary-treasurer.....Geraldine Messick '27
Chairman of program committee.....Madeline Winthrup '25
Faculty Adviser.....Professor C. J. Rees.

The Ibis Club holds regular meetings every two weeks. The general subject for the year 1924–1925 was "The history of mathematics."

The calendar for the year was as follows:

December 22, 1924. "History of mathematics in Delaware College" by Dr. Harter.

January 16, 1925. "Origin of mathematical symbols" by Tacy Hurst.

February 19. "Nichomachus of Gerasa" by Florence Wilson.

March 5. "The Rhind papyrus" by Frances Goldstein.

March 19. "Isaac Newton" by Francis Ecbert.

April 2. "Felix Klein" by Sara Slaughter.

April 16. "The school mystic and its miraculous master" by Merrel Pyle.

May 3. Annual banquet. Speaker from the University of Pennsylvania.

May 21. "Tartaglia and Cardan" by Mary L. Marvel.

At each meeting the president introduced a mathematical recreation which followed the discussion of the prepared lectures.

(Report by Miss Marvel)

The Mathematics Club of Hood College, Frederick, Maryland

[*1924*, 399]

The officers for 1924–1925 were the following:

President.....Irene Sauserman '25
Secretary-treasurer.....Lulu Brant '26
Reporter.....Evelyn Ditto '26.

The program for the year was as follows:

October 13, 1924. Initiation of new members. Address of welcome by Irene Sauserman '25, president.

November 10. "The theory of relativity" by Evelyn P. Wiggin, instructor.

December 8. Debate. Resolved that the subject matter of the course in mathematics in the elementary and secondary schools of the United States is better adapted to the needs of the pupil than the corresponding courses in foreign countries.

Affirmative: Flora Morton '25, Catherine Hess '26, Lottie Yohn '27.

Negative: Beatrice Hood '26, Effie Holstein '26, Louise Wright '26.

January 12, 1925. "The aims and value of mathematics" by Ruth Reiver '25. "Cancellation of nines" by Hilda Rickard '25.

February 9. "Eclipses" by Professor Lillian O. Brown.

March 9. "Flatland" by Mary H. Moomau '25.

March 30. "The Calendar" by Grace Brown '25.

April 11. "Napier" by Dorothy Kieny '27. "Drawing animals by means of coordinates" by Anna L. Schaidt '27.

May 10. Annual picnic. (Report by Professor Packer)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3181. Proposed by C. K. Robbins, Purdue University.

Solve each of the following differential equations:

(1) $(p-ay)^a = c(p-by)^b$;

(2) $p-ay = ce^{ay/(p-ay)}$;

(3) $p^2-2apy+(a^2+b^2)y^2e^{(2a/b)\tan^{-1}(p-ay)/by} = c$,

where $p=dy/dx$ and a, b, c are given constants.

3182. Proposed by D. H. Lehmer, University of California.

Prove the following two theorems and show how they may be used to advantage in finding the factors of R.

THEOREM 1. Let R be a non-square integer of the form $8n+k$ and represent by $2^\alpha(2m+1)$ any even denominator of a complete quotient occurring in the expansion of $\sqrt{R}$ in a continued fraction. Then, if $K=1$, $\alpha \geqq 3$; if $K=4$ or 0, $\alpha \geqq 2$; if $K=5$, $\alpha \geqq 2$; and if $K=2$, 3, 6, or 7, $\alpha=1$.

THEOREM 2. If R contains a square factor, C^2, then every multiple of C appearing as a denominator of a complete quotient must also contain a factor C^2.

3183. Proposed by Nathan Altshiller-Court, University of Oklahoma.

One of two given circles is fixed, the other rolls on a fixed straight line. Find (1) the locus of the trace of the radical axis of the two circles on their line of centers; (2) the envelope of the radical axis; (3) the locus of the limiting points of the two circles.

3184. Proposed by Otto Dunkel, Washington University.

Reduce to a product the determinant

$$\begin{vmatrix} \cdots & 1 & n & \cdots & n^{p_i} & \cdots \\ \cdots & a_i & (n+1)a_i & \cdots & (n+1)^{p_i}a_i & \cdots \\ \cdots & a_i^2 & (n+2)a_i^2 & \cdots & (n+2)^{p_i}a_i^2 & \cdots \\ & \vdots & \vdots & & \vdots & \\ \cdots & a_i^{r-1} & (n+r-1)a_i^{r-1} & \cdots & (n+r-1)^{p_i}a_i^{r-1} & \cdots \end{vmatrix},$$

where the ith group of p_i+1 consecutive columns are indicated, where there are k such groups such that $p_1+p_2+\cdots+p_k+k=r$ and where n is any number, which may, in fact, have a value in any group of columns different from that in any other group of columns without affecting the result.

3185. Proposed by B. C. Wong, Berkeley, California.

Prove geometrically that the tangent to the cardioid $\rho=2a(1-\cos\theta)$ and the bisector of the vectorial angle of the point of contact meet on the cissoid $\rho=2a\sin\theta\tan\theta$.

3186. Proposed by A. A. Bennett, Lehigh University.

Let the twelve letters, $a, b, c, \ldots k, l$, denote in some unknown order, the twelve distinct residue classes taken modulo 12. Assume that the complete multiplication of 144 products, giving each product as a letter of the set, is known, but nothing further is given concerning the sum or difference of two letters. Show that each residue class is completely identifiable. Determine whether or not a like theorem holds for the case when the modulus is 16, 18, or 24.

SOLUTIONS

501 [1916, 341]. Proposed by R. P. Baker, University of Iowa.

Find the minimum amount of lumber one inch thick required to pack a gross of spheres three inches in diameter in a rectangular box.

Solution by Norman Anning, University of Michigan.

Assume that the spheres are laid in r layers, each layer containing q rows and each row containing p spheres and let the rows and layers be "staggered" so that, in places remote from the boundary, each sphere touches 12 others. That this arrangement of uniform spheres yields maximum economy of space is well known. It is assumed moreover that $p\geqq 2, q\geqq 2, r\geqq 2$.

Consider four equal spheres each of which touches the other three; their centers will be the vertices of a regular tetrahedron. Let a be the edge of this tetrahedron; b, the altitude of any face; c, the altitude of the tetrahedron. Then a is the distance between adjacent spheres in a row, b is the distance between adjacent rows in a layer, and c is the distance between adjacent layers. When $a=3$, $b=a(\sqrt{3}/2)=2.59808$, $c=a(\sqrt{6}/3)=2.44949$.

A study of plan and elevation reveals the fact that, in order to allow for the staggering mentioned above, the inside dimensions of a box must be $a+(p-1)a+(a/2)$, $a+(q-1)b+(b/3)$, $a+(r-1)c$. Our problem becomes the following: to choose positive integers p, q, r, subject to the relation $pqr\geqq 144$, such that $(3p+1.5+2)(2.59808q+1.26795+2)(2.44949r+0.55051+2)-(3p+1.5)(2.59808q+1.26795)(2.44949r+0.55051)$ shall be a minimum. The sign "$\geqq$" is used above rather than "$=$" because there is no *a priori* reason why a box designed for 150 spheres should not require less material than one designed for just 144.

It is found by trial that the minimum occurs when we put $p=4, q=6, r=6$. The minimum amount of lumber is 1571.3 cubic inches.

By-product: when $p=5, q=5, r=6$, the amount of wood required is 1600.5 cubic inches; thus a 2% increase in the amount of material used permits a 4% increase in the capacity of the box.

2694 [1918, 170]. Proposed by N. P. Pandya, Sojitra, India.

Find the locus of the centroid of a triangle, whose vertex lies on a given parabola, whose base of given length is a segment of a given straight line of unlimited length, and one of whose base angles is known.

Solution by Harry Langman, Brooklyn, New York.

Let the given side $BA=c$ lie upon the fixed line L, and let the variable triangle have the fixed angle BAP. Take L with the positive sense BA as the x axis and a line parallel to AP as the y axis. If the coördinates of P, a point on the curve, are (x, y) and the coördinates of the centroid of BAP are (x', y') then

$$x=x'+\frac{c}{3}, \qquad y=3y'.$$

If the given conic has the equation

$$lx^2+2mxy+ny^2+2l'x+2m'y+n'=0\,,$$

it is an ellipse, hyperbola, or parabola according as $ln-m^2$ is positive, negative, or zero. This property is unchanged by the above substitution, and hence the locus is the same kind of conic as the given conic. These facts also follow directly from the form of the transformation by considering its geometric significance.

2770 [1919, 171]. Proposed by A. M. Harding, University of Arkansas.

Solve the differential equation

$$\frac{d^3x}{dt^3}-2(\mu t+\lambda)\frac{dx}{dt}+(1-\mu)x=0$$

where μ and λ are constants.

Solution by R. H. Sciobereti, Berkeley, California.

This differential equation may be replaced by the following system

$$\frac{dx}{dt}=y\,;\qquad \frac{dy}{dt}=z\,;\qquad \frac{dz}{dt}=2(\mu t+\lambda)y+(\mu-1)x\,. \tag{1}$$

For $\mu=0$, it reduces to

$$\frac{dx}{y}=\frac{dy}{z}=\frac{dz}{2\lambda y-x}=dt\,, \tag{2}$$

whose general integral is

$$\left.\begin{aligned} x&=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}+C_3e^{\lambda_3 t}\\ y&=C_1\lambda_1e^{\lambda_1 t}+C_2\lambda_2e^{\lambda_2 t}+C_3\lambda_3e^{\lambda_3 t}\\ z&=C_1\lambda_1^2e^{\lambda_1 t}+C_2\lambda_2^2e^{\lambda_2 t}+C_3\lambda_3^2e^{\lambda_3 t}\end{aligned}\right\}, \tag{3}$$

where C_1, C_2, C_3 are three arbitrary constants and λ_1, λ_2, λ_3 the three roots of the equation $\alpha^3-2\lambda\alpha+1=0$, if these roots are all distinct. If we now consider the integral which reduces to x_0, y_0, z_0 for $t=0$, then the three constants C_i will be determined by the three equations:

$$x_0=C_1^0+C_2^0+C_3^0\,;\qquad y_0=\lambda_1C_1^0+\lambda_2C_2^0+\lambda_3C_3^0\,;\qquad z_0=\lambda_1^2C_1^0+\lambda_2^2C_2^0+\lambda_3^2C_3^0\,.$$

It will be possible to vary these arbitrary constants in such a way that equations (3) will be a solution of system (1). These new variables $C_i(t)$ must however satisfy the two following conditions: for $t=0$, they must reduce to C_1^0, C_2^0, C_3^0 respectively, and satisfy the differential equations obtained by the substitution of (3) in the original system (1); we thus find:

$$\frac{\partial x}{\partial t}+\sum_i\frac{\partial x}{\partial C_i}\frac{dC_i}{dt}=\sum_i C_i\lambda_ie^{\lambda_i t}$$

$$\frac{\partial y}{\partial t}+\sum_i\frac{\partial y}{\partial C_i}\frac{dC_i}{dt}=\sum_i C_i\lambda_i^2e^{\lambda_i t}$$

$$\frac{\partial z}{\partial t}+\sum_i\frac{\partial z}{\partial C_i}\frac{dC_i}{dt}=(2\mu t+\lambda)\sum_i C_i\lambda_ie^{\lambda_i t}+(\mu-1)\sum_i C_ie^{\lambda_i t},$$

which by virtue of (3) reduce to the following:

$$\sum_i e^{\lambda_i t}\frac{dC_i}{dt}=0\,,\qquad \sum_i e^{\lambda_i t}\lambda_i\frac{dC_i}{dt}=0\,,\qquad \sum_i e^{\lambda_i t}\lambda_i^2\frac{dC_i}{dt}=\mu\sum_i C_ie^{\lambda_i t}(2\lambda_it+1)\,.$$

Solving these with respect to the derivatives, we shall obtain

$$\frac{dC_k}{dt}=\beta_k\mu\,e^{-\lambda_k t}\sum_i C_ie^{\lambda_i t}(2\lambda_it+1)\,,\qquad (k=1,2,3) \tag{4}$$

where the β_k have the following constant values:

$$\beta_1=\frac{1}{(\lambda_1-\lambda_2)((\lambda_1-\lambda_3)},\quad \beta_2=\frac{1}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)},\quad \beta_3=\frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)},$$

The system of differential equations (4) has the following properties:
i) for $\mu=0$, its integral reduces to $C_k=C_k^0$,
ii) the functions in the right members having finite partial derivatives with respect to C_i and μ satisfy Lipschitz's condition,
iii) the coefficients of the C_i are integral functions of t and of the parameter μ.
Consequently this system may be integrated by Picard's method of successive approximations, the integrals being themselves integral functions of μ.
From (4) we derive the three integral equations

$$C_k=C_k^0+\beta_k\mu\int_0^t[e^{-\lambda_k t}\sum_i C_i e^{\lambda_i t}(2\lambda_i t+1)]dt \qquad (k=1,2,3)$$

in the right members of which the particular solutions $C_k=C_k^0$ are substituted as a first approximation; hence

$$C_k^1=C_k^0+\beta_k\mu\int_0^t[e^{-\lambda_k t}\sum_i C_i^0 e^{\lambda_i t}(2\lambda_i t+1)]dt\ ;$$

for a second approximation we put C_k^1 under the integral sign and repeat this operation n times, obtaining in this manner the following expression:

$$C_k^n=C_k^0+\beta_k\mu\int_0^t[e^{-\lambda_k t}\sum_i C_i^{n-1} e^{\lambda_i t}(2\lambda_i t+1)]dt\ .$$

Now when we let n approach ∞, C_k^n will converge toward limits $C_k(t,\mu)$ which form the integral of system (4), reducing to C_k^0 for $t=0$.

REMARK.—It is also possible to derive the integrals of (1), which will take given initial values by a direct substitution in (1) of series of the form

$$\left.\begin{aligned} x&=x_0(t)+\sum_k x_k(t)\mu^k\\ y&=y_0(t)+\sum_k y_k(t)\mu^k\\ z&=z_0(t)+\sum_k z_k(t)\mu^k \end{aligned}\right\}. \qquad (5)$$

The functions $x_0(t)$, $y_0(t)$ and $z_0(t)$ must take the given initial values for $t=0$, while $x_k(t)$, $y_k(t)$, $z_k(t)$, $(k\geqq 1)$ must vanish for $t=0$. The coefficients of (5) are determined by systems of linear differential equations. Furthermore, since the coefficients of (1) are also linear in μ, an auxiliary system of the form

$$\frac{dX}{dt}=\frac{dY}{dt}=\frac{dZ}{dt}=(A+B\mu)(X+Y+Z)$$

may be chosen in such a way that it will be dominant for the given system, the quantities A and B being positive constants. The integrals of this auxiliary system which reduce to x_0, y_0, z_0 for $t=0$, are integral functions of the parameter μ, and consequently the integrals of the proposed system (1) will also be integral functions of μ. (*cf.* H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. I, p. 58 *et seq.*)

NOTE: If the roots of the equation $\alpha^3-2\lambda\alpha+1=0$ are not all distinct, equations (3) must be replaced by a different set and the solution must be modified accordingly.

2784 [1919, 366]. Proposed by T. H. Gronwall, New York City.

Show that all solutions in integers of $y^2=1+x+x^2+x^3+x^4$ are given by

$$x=-1,\quad 0,\quad 3\ ;$$
$$y=\pm 1,\pm 1,\pm 11\ .$$

Solution by A. A. Bennett, Lehigh University.

The given equation may be written in two ways

$$\left[x^2+\frac{x}{2}+\frac{\sqrt{5}-1}{4}\right]^2=y^2-\frac{(5-2\sqrt{5})}{4}\left[x+\frac{3+\sqrt{5}}{2}\right]^2, \tag{1}$$

$$\left[x^2+\frac{x}{2}+1\right]^2=y^2+\frac{5x^2}{4}. \tag{2}$$

Hence if x and y have any real values satisfying the given equation we must have

$$x^2+\frac{x}{2}+\frac{\sqrt{5}-1}{4} \leqq |y| \leqq x^2+\frac{x}{2}+1, \tag{3}$$

and hence

$$|y|=x^2+\frac{x+a}{2}, \qquad 0<a\leqq 2,$$

If x and y are required to be integers, a must be 2 if x is even, and in this case (2) shows that $x=0$. If x is odd then a must be unity. In this case, since

$$y^2=\left[x^2+\frac{x+1}{2}\right]^2-\frac{(x-3)(x+1)}{4},$$

we must have $x=3$ or $x=-1$. This gives the results stated in the problem.

2873. [1921, 36–1925, 520]. Proposed by D. H. Richert, Bethel College, Newton, Kan.

At B is the enemy's battery. At M_1 a battery is to be placed to silence B. Listening posts are installed at M_1, M_2, M_3, all provided with stop-watches. From the maps at hand, the three sides of the triangle $M_1M_2M_3$ are known. B is not visible from any one of the points M_1, M_2, M_3. The sound of a gun fired at B reaches M_1 at the time T, and M_2 at the time $T+\tau_1$ sec., and it reaches M_3 at the time $T+\tau_2$ sec. How far is B from M_1?

Solution by Norman Anning, University of Michigan.

Take as unit of distance the distance travelled by sound in air in one second at the time and place in question and take as zero of time the instant at which the gun was fired. Then $BM_1=T$; $BM_2=T+\tau_1$; $BM_3=T+\tau_2$. It is assumed that B is in the plane of the triangle $M_1M_2M_3$ and that the known distances M_2M_3, M_3M_1, M_1M_2 are respectively t_1, t_2, t_3.

Connecting the six distances of the four coplanar points there exists the identical relation:

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & T^2 & (T+\tau_1)^2 & (T+\tau_2)^2 \\ 1 & T^2 & 0 & t_3^2 & t_2^2 \\ 1 & (T+\tau_1)^2 & t_3^2 & 0 & t_1^2 \\ 1 & (T+\tau_2)^2 & t_2^2 & t_1^2 & 0 \end{vmatrix}=0.$$

From this equation, the value of T can be found in terms of the quantities τ_1, τ_2, t_1, t_2, t_3, which are all known.

Lest the enemy should get the range of M_1 while the above equation is being solved it is better to proceed as follows:

By means of the string construction, describe on the map an arc of the hyperbola in which P moves when $PM_2-PM_1=\tau_1$; likewise an arc of the hyperbola for which $PM_3-PM_1=\tau_2$; and, as a check, an arc of the hyperbola for which $PM_3-PM_2=\tau_2-\tau_1$. Under ideal conditions of observation and measurement, these three curves will be concurrent at the map position of B. Under fire, they will be nearly concurrent and will indicate the approximate position of B. Give to an artillery officer at M_1 the map position of B; the required result will follow.

Salmon states (*Conic Sections*, p. 134) that the identity quoted above is due to Cayley.

NOTE BY OTTO DUNKEL. A geometrical construction may be made to obtain the position of B. The same notation as above will be used. Draw the circles with centers M_2, M_3, and radii τ_1, τ_2; then B is the center of a circle tangent externally to the circles at P_2, P_3, respectively. It is easily shown that P_2P_3 passes through a center of similitude S of the circles M_2 and M_3; and that if Q_2 and Q_3 are the other two points of intersection, M_2Q_2, M_2P_3 are parallel as also M_2P_2, M_3Q_3. Let the tangents from S to the circles M_2, M_3, B be SN_2, SN_3, SB_1; then it may be shown that $SB_1^2=SN_2.SN_3$. Let the line SM_1 cut the circle B again at M, then a circle C passed through N_2, N_3, and M_1 cuts SM_1 in M, which may be thus found. Let the radical axis of C and M_2 cut SM_1 in R_2, and, similarly, let the radical axis of C and M_3 cut SM_1 in R_3. Then the tangent from R_2 to M_2 touches the circle at P_2, likewise the tangent from R_3 to M_3 touches at P_3. We have now merely to draw M_2P_2 and M_3P_3 in order to locate B as their intersection.

2938. [1921, 467]. Proposed by C. F. Gummer, Queen's University.

If $a, b, \ldots, i$ are real numbers $\geqq 0$, and if $\begin{vmatrix} a^r & b^r & c^r \\ d^r & e^r & f^r \\ g^r & h^r & i^r \end{vmatrix}$ is equal to zero for five real values of r other than zero, prove that the determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ has either two rows or two columns proportional, or a single row or column of zeros.

SOLUTION BY OTTO DUNKEL, Washington University.

Set the first determinant of the problem equal to $D(r)$, then the second determinant is $D(1)$. If $D(1)$ has a column or row of zeros the theorem is true. Suppose next that it has a column or row with two zeros and the third element is not zero, then by interchange of columns and rows this column or row may be made the first one with the non-vanishing element in the upper left hand corner, i.e. $a\neq 0$. Hence in this case $D(r)=a^r[(ei)^r-(hf)^r]$, and since it is zero for a value of $r\neq 0$, $ei-hf=0$. In this case the theorem is again true.

We may now assume that no two elements of a column or row are zero. If only one element of $D(1)$ is zero we may suppose it to be a. If two elements are zero we may suppose that one of these elements is a, and in this case no one of the elements b, c, d, g can be zero. Hence the second element must be e, f, h, or i. By suitable changes this zero element can be brought to the position of e, and so we shall suppose it to be e. In this case neither f nor h can be zero. Finally if there is a third zero element it must be i if the other two zeros are placed as before. With this understanding of the position of the zeros

$$D(r)=A_1^r+A_2^r+A_3^r-A_4^r-A_5^r-A_6^r \tag{1}$$

where

$$\begin{aligned} A_1&=aei\,, & A_4&=ahf\,,\\ A_2&=dhc\,, & A_5&=dbi\,,\\ A_3&=gbf\,, & A_6&=gec\,. \end{aligned} \tag{2}$$

We have the following cases to consider: I. no one of the six terms (2) is zero; II. the only terms of (2) which are zero are A_1 and A_4; III. the only zero terms are A_1, A_4, and A_6; IV. the only zero terms are A_1, A_4, A_5, A_6. This last case must be excluded since A_2 and A_3 have the same sign and $D(r)$ cannot vanish.

It will be convenient now to prove the

LEMMA. *If $A_1, A_2, \cdots, A_k$ are real quantities greater than zero, and $c_1, c_2, \cdots, c_k$ are also real and no one of them is zero, and if*

$$c_1A_1^x+c_2A_2^x+\cdots+c_kA_k^x \tag{3}$$

vanishes for k finite values of x, then $A_1=A_2=\cdots=A_k$.

PROOF. Suppose that (3) is zero for k values of x, then the derivative of

$$c_1+c_2\left(\frac{A_2}{A_1}\right)^x+\cdots+c_k\left(\frac{A_k}{A_1}\right)^x$$

vanishes for $k-1$ values of x. Thus

$$c_2\log\left(\frac{A_2}{A_1}\right)+c_3\log\left(\frac{A_3}{A_1}\right)\left(\frac{A_3}{A_2}\right)^x+\cdots+c_k\log\left(\frac{A_k}{A_1}\right)\left(\frac{A_k}{A_2}\right)^x$$

vanishes for $k-1$ values of x. Then by the same reasoning

$$c_3\log\left(\frac{A_3}{A_1}\right)\log\left(\frac{A_3}{A_2}\right)+c_4\log\left(\frac{A_4}{A_1}\right)\log\left(\frac{A_4}{A_2}\right)\left(\frac{A_4}{A_3}\right)^x+$$
$$\cdots+c_k\log\left(\frac{A_k}{A_1}\right)\log\left(\frac{A_k}{A_2}\right)\left(\frac{A_k}{A_3}\right)^x$$

vanishes for $k-2$ values of x. Finally

$$c_k\log\left(\frac{A_k}{A_1}\right)\log\left(\frac{A_k}{A_2}\right)\cdots\log\left(\frac{A_k}{A_{k-1}}\right)$$

vanishes for one value of x. Hence at least one pair of the A's are equal. The terms of (3) having equal A's can be combined and we have an expression of form (3) with fewer than k terms which vanishes for k distinct values of x. By repetition of this process we finally find that, if (3) vanishes for k distinct values of x, the A's must all be equal.

Now in case I, $D(0)=0$ and hence $D(r)$ vanishes for six values of r and it follows from the lemma that all of the A's in (2) are equal. This says that $a(ei-hf)=0$ etc. and hence the second and third columns are proportional to the first. In case II we have

$$A_2^r+A_3^r-A_5^r-A_6^r$$

is zero for four values of r, and hence $a=0$, $bf-ec=hc-bi=0$. Thus the last two columns are proportional. In case III we have $A_2^r+A_3^r-A_4^r$ which by the lemma can vanish for three values of r only if $A_2=A_3=A_5=0$, which is contrary to hypothesis. This completes the proof of the theorem of the problem.

3131. [1925, 204]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Find the locus of the mid-point of the segment intercepted by two fixed tangents to a given conic on a variable tangent to the same conic.

SOLUTION BY T. W. EDMONDSON, New York University

Taking the two fixed tangents as axes of coördinates, the equation of the conic may be written

$$g^2x^2+2chxy+f^2y^2+2cgx+2cfy+c^2=0\ .$$

Let the variable tangent be $lx+my+n=0$. Then the tangential equation of the conic is

$$\begin{vmatrix} g^2 & ch & cg & l \\ ch & f^2 & cf & m \\ cg & cf & c^2 & n \\ l & m & n & 0 \end{vmatrix}=0\ ,$$

which reduces to

$$(fg+ch)n^2-2cgmn-2cfnl+2c^2lm=0\ ,$$

if we remove the factor $(ch-fg)$, which is not zero since the conic is assumed to be non-degenerate.

The coördinates of the mid-point of the segment intercepted on the variable tangent are

$$x=-\frac{n}{2l}\ ,\ y=-\frac{n}{2m}$$

whence

$$l=-\frac{n}{2x}, \qquad m=-\frac{n}{2y}.$$

Substituting these values in the tangential equation, we find that the equation of the locus of the mid-point is

$$2(fg+ch)xy+2cgx+2cfy+c^2=0,$$

a hyperbola whose asymptotes are parallel to the two fixed tangents.

Also solved by MICHAEL GOLDBERG, M. S. KNEBELMAN, F. H. LOUD, and AUGUST SÖRENSEN.

3132. [1925, 204]. Proposed by J. W. Clawson, Ursinus College.

The trilinear coördinates of the circumcenter of a triangle are: cos A, cos B, cos C; that is, the distances of this point from the sides of the triangle are proportional to cos A, cos B, and cos C respectively. The points (sin A, sin B, sin C), (sec A, sec B, sec C), and (csc A, csc B, csc$C$) are also well-known points (symmedian, orthocenter, and centroid) which can be located by simple geometrical constructions.

Can the points (tan A, tan B, tan C) and (cot A, cot B, cot C) be found by simple geometrical constructions? Have the points been named?

SOLUTION BY MICHAEL GOLDBERG, Philadelphia, Penn.

The point (tan A, tan B, tan C) may be constructed as follows: Draw a line parallel to AB at a distance tan C from it. To obtain tan C lay off along one side of the angle C an arbitrary unit length and erect a perpendicular at its end to meet the other side. Draw a line parallel to BC at a distance tan A intersecting the first parallel at D; similarly, a parallel to AC intersecting the first parallel at E. The desired point is the intersection of AE and BD.

The construction for (cot A, cot B, cot C) is similar. I am not aware that these points have special names.

3135 [1925, 261]. Proposed by N. P. Pandya, Amreli, Kathiawad, India.

Form the equation of lowest degree with rational coefficients whose roots are sin A, sin $4A$, sin $7A$, and sin $10A$, where $13A=\pi$.

SOLUTION BY A. G. CLARK, Colorado Agricultural College.

The twelve imaginary roots of unity satisfy the equation

$$x^{12}+x^{11}+\cdots+x+1=0, \tag{1}$$

Setting $y=x+x^{-1}$ we obtain the equation

$$y^6+y^5-5y^4-4y^3+6y^2+3y-1=0, \tag{2}$$

whose roots are 2 cos $2nA$, $n=1, 2, 3, 4, 5, 6$, $A=\pi/13$. Since $2\cos 2nA=2-4\sin^2 nA$, if we reduce the roots of (2) by 2, and divide the roots of the resulting equation by -4, we obtain the equation for $\sin^2 nA$. The equation $4096z^6-13312z^5+16640z^4-9984z^3+2912z^2-364z+13=0$, is satisfied by $\sin^2 A$, $\sin^2 4A$, $\sin^2 7A=\sin^2 6A$, $\sin^2 10A=\sin^2 3A$, $\sin^2 2A$, $\sin^2 5A$.

NOTE BY OTTO DUNKEL, Washington University.

By a method given by Gauss the equation (2) may be solved by solving first the cubic

$$x^3+x^2-4x+1=0, \tag{4}$$

all of whose roots are real, and then by solving quadratics of the form

$$x^2-x_1x+x_2=0,$$

where x_1 is the smaller positive root and x_3 the negative root of (4) [see Burnside and Panton's *Theory of Equations*, vol. 1, ex. 15, page 101].

The equation (2) may also be reduced to two cubics

$$\begin{aligned} 2y^3+(1-\sqrt{13})\,y^2-2y-3+\sqrt{13}=0\,, \\ 2y^3+(1+\sqrt{13})\,y^2-2y-3-\sqrt{13}=0\,. \end{aligned} \tag{5}$$

This may be verified by taking the product of the left sides of the two equations. The roots of the first are $2\cos 2A$, $-2\cos 5A$, $2\cos 6A$; the roots of the second are $2\cos 4A$, $-2\cos 3A$, $-2\cos A$.

It should be observed that in the above solution it has not been shown that there is no equation of lower degree having the desired roots.

Also solved by MICHAEL GOLDBERG.

3136 [1925, 261]. Proposed by Paul Capron, U. S. Naval Academy.

If the sines of the half sides of a spherical triangle are h, k, l, and the cosines of the halves of the corresponding opposite angles are H, K, L, and if P and p are the polar distances of the circumscribed and inscribed circles then

$$\sin P=\frac{hkl}{2\sqrt{\sigma(\sigma-h)(\sigma-k)(\sigma-l)}}\,, \qquad \cos p=\frac{HKL}{2\sqrt{\Sigma(\Sigma-H)(\Sigma-K)(\Sigma-L)}}\,,$$

where $\sigma=(h+k+l)/2$ and $\Sigma=(H+K+L)/2$.

SOLUTION BY R. H. SCIOBERETI, Berkeley, California.

Let us start from the two fundamental formulae

$$\tan P=\sqrt{\frac{\sin S}{\sin(A-S)\sin(B-S)\sin(C-S)}}=\frac{\sin S}{N} \tag{1}$$

and

$$\cos P=\sqrt{\frac{\tan(A-S)\tan(B-S)\tan(C-S)}{\tan(A-S)+\tan(B-S)+\tan(C-S)}} \tag{2}$$

where $2S$ represents the spherical excess, and

$$N=\sqrt{\sin S\sin(A-S)\sin(B-S)\sin(C-S)}, \quad \text{(norm of the angles)}$$

$$n=\sqrt{\sin s\sin(s-a)\sin(s-b)\sin(s-c)}, \quad \text{(norm of the sides)},$$

$$s=(a+b+c)/2\,.$$

We shall express all the trigonometric functions of the preceding angles by means of the trigonometric functions of the sides. From

$$\tan\tfrac{1}{2}a=\sqrt{\frac{\sin S\ \sin(A-S)}{\sin(B-S)\sin(C-S)}}\,, \qquad \tan\tfrac{1}{2}b=\sqrt{\frac{\sin S\ \sin(B-S)}{\sin(C-S)\sin(A-S)}}\,, \text{ and}$$

$$\tan\tfrac{1}{2}c=\sqrt{\frac{\sin S\sin(C-S)}{\sin(A-S)\sin(B-S)}}\,,$$

we obtain by multiplication $\tan\tfrac{1}{2}a\cdot\tan\tfrac{1}{2}b\cdot\tan\tfrac{1}{2}c=\dfrac{\sin^2 S}{N}$,

and by comparison with equation (1) $\tan P=\dfrac{\tan\tfrac{1}{2}a\cdot\tan\tfrac{1}{2}b\cdot\tan\tfrac{1}{2}c}{\sin S}$.

Now from Cagnoli's formula giving the spherical excess as a function of the sides

$$\sin S=\frac{n}{2\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c} \tag{4}$$

we easily derive the following expressions:

$$\left.\begin{aligned}\sin(A-S)&=\frac{n}{2\cos\tfrac{1}{2}a\sin\tfrac{1}{2}b\sin\tfrac{1}{2}c}\\ \cos(A-S)&=\frac{\sin^2\tfrac{1}{2}b+\sin^2\tfrac{1}{2}c-\sin^2\tfrac{1}{2}a}{2\cos\tfrac{1}{2}a\sin\tfrac{1}{2}b\sin\tfrac{1}{2}c}\end{aligned}\right\}$$

and similar relations for $(B-S)$, $(C-S)$; hence

$$\tan(A-S)=\frac{n}{\sin^2\tfrac{1}{2}b+\sin^2\tfrac{1}{2}c-\sin^2\tfrac{1}{2}a}=\frac{n}{k^2+l^2-h^2},$$

$$\tan(B-S)=\frac{n}{l^2+h^2-k^2},\qquad \tan(C-S)=\frac{n}{h^2+k^2-l^2}.$$

Substituting these values in equation (2), we shall have

$$\cos P=\sqrt{\frac{n^3}{(k^2+l^2-h^2)(l^2+h^2-k^2)(h^2+k^2-l^2)}\Big/ n\left[\frac{1}{k^2+l^2-h^2}+\frac{1}{l^2+h^2-k^2}+\frac{1}{h^2+k^2-l^2}\right]},\tag{5}$$

$$\tan P=\frac{2\sin\tfrac{1}{2}a\sin\tfrac{1}{2}b\sin\tfrac{1}{2}c}{n}=\frac{2hkl}{n}.\tag{6}$$

Finally if we multiply (5) by (6), we shall obtain the required formula

$$\sin P=\frac{2hkl}{n}\cdot\frac{n\left[(h^2+k^2-l^2)(k^2+l^2-h^2)(l^2+h^2-k^2)\right]^{-1/2}}{\left[\dfrac{1}{k^2+l^2-h^2}+\dfrac{1}{l^2+h^2-k^2}+\dfrac{1}{h^2+k^2-l^2}\right]^{1/2}},$$

$$\sin P=2\,hkl\,[(h^2+k^2-l^2)(l^2+h^2-k^2)+(k^2+l^2-h^2)(h^2+k^2-l^2)+(l^2+h^2-k^2)(k^2+l^2-h^2)]^{-1/2}.$$

The second factor may be written after a few simple transformations

$$\sqrt{4h^2k^2-(h^2+k^2-l^2)^2}=\sqrt{(h+k+l)(k+l-h)(l+h-k)(h+k-l)};$$

hence

$$\sin P=\frac{hkl}{2\sqrt{\sigma(\sigma-h)(\sigma-k)(\sigma-l)}}.$$

Note by Otto Dunkel, Washington University.

The two formulae of the problem are consequences of a relation which may be stated thus: If x, y, z are three angles such that the sine of one angle is equal to the sine of the sum of the other two, then

$$4s(s-\sin x)(s-\sin y)(s-\sin z)=\sin^2x\sin^2y\sin^2z,\tag{1}$$

where $2s=\sin x+\sin y+\sin z$.

If O is the center of the circumscribing circle of the triangle ABC and if A', B', C' are the mid-points of the sides, then, if we set $\angle BOA'=x$, $\angle COB'=y$, $\angle AOC'=z$, either the sum of two of the angles is equal to the third or $x+y+z=180°$. Now from the right triangles such as BOA' we have $\sin x=h/\sin P$, and then $s=\sigma/\sin P$, $s-\sin x=(\sigma-h)/\sin P$. The first formula follows by substituting these in (1).

If O is the center of the inscribed circle and A', B', C' are the feet of perpendiculars from O to the sides, we may set $x=\angle BOA'$, $y=\angle COB'$, $z=\angle AOC'$. (These letters have not, of course, the same meanings as in the preceding paragraph.) Here $x+y+z=180°$, and from one of the right triangles such as BOA' we have $\sin x=K/\cos p$, $s=\Sigma/\cos p$, $s-\sin x=(\Sigma-K)/\cos p$, and the second formula follows by substitution in (1). In either case the expression under the radical is easily seen to be positive and the positive sign is to be taken for the radical.

Also solved by J. A. Bullard and A. Pelletier.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

At its meeting in April, 1926, the National Academy of Sciences elected as foreign associates Professor JACQUES HADAMARD of the Ecole Polytechnique and the Sorbonne, and Professor MAX PLANCK, director of the Institute of Theoretical Physics of the University of Berlin. Professor OSWALD VEBLEN was elected to the Council of the Academy for a period of three years.

Harvard University has awarded from its Milton Fund for Research a sum of money to Professor E. B. WILSON for expenses connected with investigations of stellar statistics to be pursued in accordance with biometric methods involving partial correlation.

At Princeton University, Professor J. W. ALEXANDER has been promoted to an associate professorship, and Dr. T. Y. THOMAS has been appointed to an assistant professorship of mathematics.

Professor E. T. BELL, of the University of Washington, has been appointed professor of mathematics at the California Institute of Technology.

Associate Professor J. D. BOND, of the University of Tennessee, has been promoted to a full professorship of mathematics.

Dr. LAURA BRANT, of Vassar College, has been appointed professor of physics and mathematics at Judson College, Marion, Ala.

Assistant Professor THOMAS BUCK, of the University of California, has been promoted to an associate professorship of mathematics.

Assistant Professor G. M. CONWELL, of the New York State College for Teachers, has been promoted to a full professorship of mathematics.

Dr. L. M. GRAVES has been appointed assistant professor of mathematics at the University of Chicago.

Assistant Professor GERTRUDE A. HERR, of Iowa State Teachers College, has been promoted to an associate professorship of mathematics.

Dr. JEWELL C. HUGHES, of the University of Arkansas, has been promoted to an assistant professorship of mathematics.

Assistant Professor GLENN JAMES has been projoted to an associate professorship of mathemathics at the Southern Branch of the University of California.

Associate Professor H. E. JORDAN, of the University of Kansas, has been promoted to an associate professorship of mathematics.

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The Association needs funds for scientific publications and for the promotion of scientific activities.

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BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Tenth Summer Meeting of the Association, Columbus, Ohio, September 7-8, 1926.

Eleventh Annual Meeting, Philadelphia, Pa., December, 30-31, 1926.

The following are dates of Section Meetings of the Association in 1926:

ILLINOIS, Decatur, Ill., May 7-8.

INDIANA, Purdue University, May, 7-8.

IOWA, Cedar Rapids, April.

KANSAS, Merged in National Meeting.

KENTUCKY, Berea College, May 1.

LOUISIANA-MISSISSIPPI, New Orleans, La., March 12-13.

MARYLAND - DISTRICT OF COLUMBIA - VIRGINIA, Baltimore, Md., December 4.

MICHIGAN, Ann Arbor, Mich., April 1.

MINNESOTA, Northfield, Minn., May 22.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Bethany, Neb., May.

OHIO, Columbus, Ohio, April 2.

ROCKY MOUNTAIN, Colorado College, April, 1927.

SOUTHEASTERN, Atlanta, Ga., March 19-20.

SOUTHERN CALIFORNIA, Los Angeles, Calif., November 6.

TEXAS, November.

Secretaries of Sections will please report changes or corrections promptly to the Editor.

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MEETING OF THE SOUTHERN CALIFORNIA SECTION

The third regular meeting of the Southern California Section of the Mathematical Association of America was held at the California Institute of Technology, in Pasadena, on Saturday, March 13, 1926, Professor Harry Bateman presiding.

There were fifty present, including the following twenty-five members of the Association: O. W. Albert, E. E. Allen, H. Bateman, F. P. Brackett, J. R. Campbell, M. Collier, M. E. Conn, P. H. Daus, H. H. Gaver, H. E. Glazier, E. R. Hedrick, H. C. Hicks, G. H. Hunt, W. E. Mason, G. F. McEwen, W. A. Newlin, B. Podolsky, W. P. Russell, H. M. Showman, M. Skarstedt, D. V. Steed, H. C. Van Buskirk, L. E. Wear, Clyde Wolfe, E. R. Worthington.

The following officers were elected for the coming year: Professor H. C. WILLETT, chairman, Professor W. P. RUSSELL, Vice-chairman, Professors E. E. ALLEN and L. E. WEAR, program committee.

The following papers were presented. Short abstracts appear below.

1. "Fibonacci series," by Mr. MORGAN WARD, California Institute of Technology (by invitation).

2. "Function space," by Professor E. R. Hedrick, University of California, Southern Branch.

3. "An example of periodic analysis of river flow," by BORIS PODOLSKY, University of Southern California.

4. "Certain link motions connected with conformal transformations," by Mr. H. C. HICKS, California Institute of Technology.

5. "On imaginary elements in construction problems," by Professor P. H. DAUS, University of California, Southern Branch.

6. "A demonstrative talk on lightning phenomena," by Professor R. W. SORENSON, California Institute of Technology (by invitation).

1. Mr. Ward considered the residues modulo p of the terms of a Fibonacci series and the cycles formed for different moduli. He discussed the generalization to series formed from the difference equation $x_n = px_{n-1} + qx_{n-2}$ and some of the resulting properties.

2. Professor Hedrick discussed the fundamental concepts of function space as developed by Hilbert, Schmidt, and Jackson and showed the relation between these ideas and our ordinary concepts of least squares, standard deviation, vector cross product and others.

3. Mr. Podolsky discussed the analysis of the flow of Owens River, by means of the apparatus he described at a previous meeting.